Coast of a fuzzy set as a 'crisper' subset of the boundary

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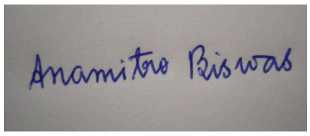

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It is certified that the work contained in this thesis entitled "**Coast of a fuzzy set as a 'crisper' subset of the boundary**" submitted by Anamitro Biswas, Scholar No. 2148106 for the award of M.Sc. in Mathematics is absolutely based on his work carried out under my supervision and that his work has not been submitted elsewhere for any degree.

Dr. Juthika Mahanta
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Abstract

Geographical information systems rely on the concept of topological spaces for analysing the level of interaction between two or more geographical entities and determining the extent of each. In real life, belonging of an element in a set is not just affirmation or denial, but rather a matter of degree. Zadeh introduced 'fuzziness' which was adapted into fuzzy topology first by Chang. Spatial relations introduced by Egenhofer in the context of application of topology in GIS can be extended in various ways between fuzzy regions. These ideas have a wide range of application in Geographical Information Systems, in forecasting floods, preparing terrain maps for defence etc. This report covers a brief introduction to fuzzy set theory followed by notions of fuzzy topology and how they can improve over application of crisp topology in GIS. Extension of concepts from general topology to fuzzy sets have been explored with an ultimate aim of discussing in brief the concepts that are prerequisite to application of fuzzy topology to GIS by finding topological invariants and spatial relations for simple fuzzy regions.

The collected geographical data is often wanted to retain its innate fuzziness for our study, nonetheless a strict boundary existing between a set and its complement, or its interior and exterior, makes things a lot easier. An example might be the making of jurisdictory maps for coastal or delta regions, where we need to demarcate between land and water and make an idea about the area under a constituency or police station. Coast of a fuzzy set has been defined, along with possible alternative definitions, which in a way is shown to solve this purpose also. Further, various properties of Coast has been studied along with its distinctness of definition and purpose from fuzzy boundary.

Chapter 1

Introduction to Fuzzy Topology

1.1 Backdrop

Heterogeneous data of varied origin and accuracy include some level of uncertainty, that can be viewed as as a measure of differences between data and measurement that is assigned by the standard user to be received in course of accurate study of reality. Therefore, there is an increasing need for information with uncertainty-based GIS. The geographical data recorded in a GIS represent some phenomena. For example, to determine the true class of the soil is a near-impossibility while the definition of classes of soil are mostly vague or inaccurate. Therefore, a more realistic view of geographical measurement than the concept of "error" is the concept of "uncertainty", whose formulation in the fuzzy form reduces the possibility of getting solutions that are incompatible in the calculation and optimization.

While trying to deal mathematically with the problems in the world, one comes across 'organized complexity' [Warren Waver, 1948] or non-linear systems with a large number of richly-interacting components, which are non-deterministic, but not as a consequnece of randomness that can justify the use of statistical averages. In the 1960s, approaches to this end began to be explored, among which L. A. Zadeh's paper is certainly worth mentioning. The idea can be traced back to 1937 by American philosopher Max Black. In that paper, Zadeh broke away from Aristotelian two-valued logic and introduced a "fuzzy" set, where membership is not just subject to of affirmation or denial, but rather is a matter of degree. Egerhofer and Franzosa [6] introduced a model of spatial relations based on point-set topological notions of *interior*, *closure* and *boundary*. An adaptation of that for geographical data represented by fuzzy regions had been proposed by Tang and Kainz [15], where the binary topological relations have been formulated as 3×3 and 4×4 matrices.

1.2 Set terminology

Definition 1.2.1. *Let Ω be a space of points where a generic element is denoted by x. A* fuzzy set (class) *$A \subseteq \Omega$ is characterized by a* membership (characteristic) function *$\mu_A : \Omega \to [0, 1]$ with the value of $\mu_A(x)$ denoting the* grade of membership *of x in A,*

$$A = \{(x, \mu_A(x)) \mid x \in \Omega\} . \tag{1.1}$$

When the membership function has value 1 or 0 this is called *crisp set* whose membership function takes only 2 values.

Definition 1.2.2. 1. *A is an* empty *fuzzy set [denoted by 0_Ω] in Ω iff $\mu_A(x) = 0 \ \forall \ x \in \Omega$.*

2. *Two fuzzy sets A and B are* equal, *i.e., $A = B$ iff $\mu_A(x) = \mu_B(x) \ \forall \ x \in \Omega$.*

3. Complement *of a fuzzy set* A *is denoted by* A^C *and is defined by* $\mu_{A^C} = 1 - \mu_A$.

4. *A fuzzy set* A *is* contained *in a fuzzy set* B *or* A *is a* subset of B *or* A *is* smaller than or equal to B *iff* $\mu_A(x) \leq \mu_B(x)\ \forall_x \in \Omega$.

Definition 1.2.3. *Let* (A, μ_A) *and* (B, μ_B) *be two fuzzy sets.*

1. *Their* union *is a fuzzy set* $C = A \cup B$ *whose membership function* μ_C *is given by* $\mu_C(x) = \max\{\mu_A(x), \mu_B(x)\}\ \ \forall\ x \in \Omega$, *or,* $\mu_C = \mu_A \vee \mu_B$. *Alternatively, The* union *of fuzzy sets* A *and* B *is the smallest fuzzy set that covers both* A *and* B.

2. Intersection $C = A \cap B$ *has membership function* $\mu_C = \min\{\mu_A, \mu_B\}$ *or,* $\mu_C = \mu_A \wedge \mu_B$. *Alternatively, the intersection* $A \cap B$ *is the largest fuzzy set which is contained in both* A *and* B.

Two fuzzy sets A and B are *disjoint* if $A \cap B$ is empty.

De Morgan's laws $[(A \cup B)^C = A^C \cap B^C$, $(A \cap B)^C = A^C \cup B^C]$ and Distributive laws $[C \cap (A \cup B) = (C \cap A) \cup (C \cap B)$, $C \cup (A \cap B) = (C \cup A) \cap (C \cup B)]$ hold for fuzzy sets. Essentially, fuzzy sets in Ω constitute a distributive lattice with a 0 and 1 [1]. *Law of Contradiction* $[A \cap A^C = 0_\Omega]$ and *Law of Excluded Middle* $[A \cup A^C = \Omega]$ do not hold for fuzzy sets since $\mu_{A \cap A^C} = \min\{\mu_A, 1 - \mu_A\} \neq 0$ and $\mu_{A \cup A^C} = \max\{\mu_A, 1 - \mu_A\} \neq 1$ for fuzzy sets.

1.3 Fuzzy sets induced by mappings

Let $R : (X, Y) \to [0, 1]$. The membership function μ_R gives the degree of relation of ordered pair $(x, y) \in (X, Y)$. Let T be a mapping from space X to space Y, and let us consider fuzzy set $B \subseteq Y$ and $A = T^{-1}(B) \subseteq X$. The membership function for B is defined by:

$$\mu_B(y) = \begin{cases} \max\limits_{x \in T^{-1}(y)} \mu_A(x), y \in Y & \text{if } T^{-1}\{y\} \neq \varnothing, \\ 0 & \text{if } T^{-1}\{y\} = \varnothing. \end{cases} \tag{1.2}$$

1.4 Partition

It is assumed from now on that Ω is a real Euclidean space E^n.

Definition 1.4.1. *The crisp set containing only elements belonging with a grade of membership of at least* α *to a fuzzy set in question, i.e.,* $\Gamma_\alpha(S) = \{x \in \Omega \mid \mu_S(x) \geq \alpha\}$. *This is also called the* weak α-cut *of* S. *The* strong α-cut *of* S *is defined as* $\sigma_\alpha(S) = \{x \in \Omega \mid \mu_S(x) > \alpha\}$. $\sigma_0(S)$ *is called the* support *of* S. *It is the crisp set of all elements of* Ω *which have non-zero membership functions with respect to* S.

Definition 1.4.2. *A fuzzy set* A *is said to be bounded iff the sets* $\Gamma_\alpha = \{x \mid \mu_A(x) \geq \alpha\}$ *are bounded in norm* $\forall\ \alpha > 0$, *i.e.,* $\forall \alpha > 0\ \exists$ *some finite* $R(\alpha) \ni \|x\| \leq R(\alpha)\ \forall\ x \in \Gamma_\alpha$.

1.5 Fuzzy Topology: Open sets, neighbourhoods and interior points

Definition 1.5.1. *Let* τ *be a family of fuzzy sets in* X *such that*

1. $0_X, 1_X \in \tau$;

2. $A, B \in \tau \implies A \cap B \in \tau$;

3. $A_i \in \tau \ \forall\ i \in I \implies \bigcup_{i \in I} A_i \in \tau.$

Then τ is called a fuzzy topology *for X and (X, τ) is called a* fuzzy topological space *or* fts *in short. Condition (1) gets redundant.*

Definition 1.5.2. *A set A is τ-open if $A \in \tau$, while it is τ-closed if and only if A^C is τ-open. [3]*

For fuzzy sets,

it makes more sense to think of neighbourhoods of a fuzzy set itself (which contains points to the extent of the values of their membership functions).

Definition 1.5.3. *A fuzzy set $U \subseteq X$ in an fts (X, τ) is said to be a* neighbourhood *(*nbhd *for short) of a fuzzy set $A \subseteq X$ if and only if there exists a τ-open fuzzy set O such that $A \subseteq O \subseteq U$. A* neighbourhood system *of a fuzzy set is the family of all its neighbourhoods.*

Definition 1.5.4. *Let A and B be fuzzy sets in an fts (X, τ) such that $A \supset B$. Then, B is an* interior fuzzy set *of A if and only if A is a nbhd of B. The* interior *of A is the union of all interior fuzzy sets of A.*

Definition 1.5.5. *The membership function of a* fuzzy point *takes value* $0 \ \forall\ y \in X$ *except one, say $x \in X$. It is denoted by x_λ $(0 < \lambda \leq 1)$, where $\mu_{x_\lambda}(x) = \lambda$, and x is called its support.*

Definition 1.5.6. *A function f from an fts (X, τ) to an fts (Y, υ) is said to be* F-continuous *if and only if $f^{-1}\left[G^{\upsilon\text{-}open}\right]$ is τ-open $\forall\ G \in \upsilon$.*

Definition 1.5.7. *A* fuzzy homeomorphism *is an F-continuous injective map from an fts X onto an fts Y, such that the inverse of the map is F-continous as well. In that case X and Y are said to be* F-homeomorphic *and each is a* fuzzy homeomorph *of the other.*

Definition 1.5.8. *[15, 2.5] A fuzzy relation is said to be a* topological relation *if it is preserved under a fuzzy homeomorphism of its embedding fts.*

1.6 Induced fuzzy topology

Let (X, τ) be a crisp topological space. Let A be a fuzzy set in X.

Definition 1.6.1. *A is closed if whenever there exists a net $(x_\alpha)_{\alpha \in \mathcal{J}} \to x \in X$, then $\mu_A(x) \geq$* $\limsup_{\alpha \in \mathcal{J}} \mu_A(x_\alpha)$.

But this is precisely the condition that the mapping $\mu_A : X \to \mathbb{R}$ is *upper semicontinuous*, i.e., $1 - \mu_A$, the membership function for "open" A^C is *lower semicontinuous.* Thus,

Definition 1.6.2. *An* induced fuzzy topology *on (X, τ) is the collection of all lower semicontinuous fuzzy sets in X. It is denoted by $F(\tau)$.*

Proposition 1.6.3. *[18, Proposition 3.2] $F(\tau)$ thus defined will be a fuzzy topology for X.*

Proposition 1.6.4 (F-continuity in induced fuzzy sets)**.** *[18, Proposition 3.4] If $(X, \tau), (Y, \upsilon)$ be topological spaces, then a mapping $T : (X, F(\tau)) \to (Y, F(\upsilon))$ is F-continuous if and only of $T : (X, \tau) \to (Y, \upsilon)$ is continuous.*

1.7 Separation and Connectedness

The fact whether two fuzzy sets are separated or connected plays a vital role in determining the extent of interaction between the respective regions in the study of geography. For that, definitions of separatedness and connectedness in crisp topology can be variously extended to fuzzy sets, as in [15].

Definition 1.7.1. *1. Fuzzy sets A, B in fts (X, τ) are* separated *iff* $\exists\ U, V \in \tau \ni U \supseteq A, V \supseteq B, U \cap B = V \cap A = 0_X$.

2. Fuzzy sets A, B in fts (X, τ) are Q-separated *iff* $\exists\ H, K \ni H^C, K^C \in \tau; H \supseteq A, K \supseteq B, H \cap B = K \cap A = 0_X$.

Definition 1.7.2. *1. A fuzzy set A is said to be* open-connected *if $\nexists$ separated $C, D \ni A = C \cup D$.*

2. A fuzzy set A is said to be closed-connected *if $\nexists$ Q-separated $C, D \ni A = C \cup D$.*

3. A fuzzy set A is said to be double-connected *if it is both open- and closed-connected.*

Chapter 2

Spatial relations and application to GIS

2.1 Spatial relations among crisp sets aided by point-set topology

Point-set topological notion of *interior* of a set A, A° refers to the collection of those points x for which $\exists\ G_x^{\text{open}} \ni x \in G_x \subseteq A$. The *closure* of A, denoted by $\overline{A}$ is the intersection of all the closed sets containing A [10]. For $A \subset X$, the *boundary* ∂A is defined as $\partial A = \overline{A} \cap \overline{X - A}$ [11].

Proposition 2.1.1. *[6, Propositions 3.1, 3.2]* $A^\circ \cap \partial A = \varnothing,\ Y^\circ \cup \partial Y = \overline{Y}$.

A set A is said to be *connected* if it is not a union of two disjoint non-empty open sets, and *separated* in the contrary case. $Z \subseteq X$ separates X is X and $X - Z$ is disconnected [6].

Proposition 2.1.2. *[6, Propositions 3.4] For $Y \subset X$ such that $Y^\circ \neq \varnothing$ and $\overline{Y} \neq X$, Y° and $X - \overline{Y}$ form a separation of $X - \partial Y$, ∂Y thus separating X.*

Let $A, B \subseteq X$. A *topological spatial relation* between A and B is described by a 4-tuple of values of *topological invariants* (i.e., preserved under homeomorphism, or continuous bijective map with continuous inverse) associated to each of the 4 sets $\partial A \cap \partial B$ (boundary-boundary intersection), $A^\circ \cap B^\circ$ (interior-interior intersection), $\partial A \cap B^\circ$ (boundary-interior intersection), $A^\circ \cap \partial B$ (interior-boundary intersection) respectively.

Definition 2.1.3. *A* spatial region *in a connected topological space X is a $(\varnothing \neq)\, A^{connected} \subset X \ni A = \overline{A^\circ}$.*

For a spatial region, $\partial A \neq \varnothing$ [6, Proposition 5.2].

The 16 exhaustive spatial relations between two sets on a plane are all feasible, but it can be proved that $(\varnothing, \varnothing, \varnothing, \varnothing)$, $(\odot, \varnothing, \varnothing, \varnothing)$, $(\odot, \odot, \varnothing, \varnothing)$, $(\varnothing, \odot, \odot, \varnothing)$, $(\odot, \odot, \odot, \varnothing)$, $(\varnothing, \odot, \varnothing, \odot)$, $(\odot, \odot, \varnothing, \odot)$, $(\varnothing, \odot, \odot, \odot)$, $(\odot, \odot, \odot, \odot)$ are possible for spatial regions [6, Proposition 5.3]. [The symbol $\odot$ stands for a non-empty set.]

2.2 Topological invariants in a fuzzy set

Due to shortcomings of the crisp model used in GIS while dealing with the phenomena of the real world, fuzzy methods have been introduced.

Definition 2.2.1. *Properties of fuzzy sets that remain invariant under fuzzy homeomorphism, are called* topological properties*.*

The following model for describing topological spatial relations between two regions represented by fuzzy sets has been based on a consideration of the intersections of *cores, fringes and outers* or *cores, b-closures, c-boundaries and outers* of two sets A and B, which are henceforth defined and proved to be topological properties.

Definition 2.2.2. *(B1) [17] The* fuzzy boundary *of a fuzzy set A is the infimum of all closed fuzzy sets D in X such that* $\mu_D(y) \geq \mu_{\overline{A}}(y)\ \forall\ y \in \left\{x \in X \mid \mu_{\overline{A}}(x) \wedge \mu_{\overline{A^C}}(x) > 0\right\}$.

(B2) [12] The fuzzy boundary *of a fuzzy set A is defined as:* $\partial A = \overline{A} \cap \overline{A^C}$.

(B2′) [2] The membership function of the fuzzy boundary ∂A *of A is defined as* $\mu_{\partial A}(x, y) = 2 \min \{\mu_A(x, y), 1 - \mu_A(x, y)\}$, *where the factor 2 normalizes the membership values.*

(B3) [5] The boundary of A is the infimum of all closed fuzzy sets B in X such that $B(x) \geq \overline{A}(x)\ \forall\ x \in X \ni \overline{A}(x) - A^\circ(x) > 0$.

(B4) [8] $\partial A = \vee \left\{x_\beta \mid \beta \leq \overline{A}(x), \beta > A \int(x)\right\}$.

(B5) [9] $\partial A = \left(\overline{A}\right)^\circ \wedge \overline{A^\circ}$.

Subsets of the boundary ∂A:[15]

- c-boundary, $\partial^c A = \{x_\lambda \in \partial A : \mu_{\partial A}(x) = \mu_{\overline{A}}(x)\}$;
- i-boundary, $\partial^i A = \{x_\lambda \in \partial A : \mu_{\partial A}(x) < \mu_{\overline{A}}(x)\}$.

Subsets of the closure $\overline{A}$:

- i-closure, $A^{\pm} = \left\{x_\lambda \in \overline{A} \mid \mu_{\partial A}(x) < \mu_{\overline{A}}(x)\right\}$;
- c-closure, $A^{\mp} = \left\{x_\lambda \in \overline{A} \mid \mu_{\partial A}(x) = \mu_{\overline{A}}(x)\right\}$.

Crisp subsets:

- Core, $A^{\oplus} = \{x_\lambda \in X \mid \mu_{A^\circ}(x) = 1\} \subseteq A^\circ$;
- Outer, $A^{=} = \{x_\lambda \in A^e \mid \mu_{A^e}(x) = 1\} \subseteq A^e$, where $A^e = \left(A^C\right)^\circ$.

Some more invariants:

- b-closure of A in the fts, $A^{\perp} = A^{\pm} - A^{\oplus}$;
- fringe of A, $\ell A = \partial^c A \cup A^{\perp}$.

Theorem 2.2.3. *[15, Propositions 4.9, 4.10]*

1. *A fuzzy homeomorphism is boundary-preserving, i.e.,* $f(\partial A) = \partial f(A)$;
2. *The following are topological properties: (i) c-boundary, (ii) i-boundary, (iii) c-closure, (iv) i-closure, (v) b-closure, (vi) core, (vii) outer.*

2.3 Fuzzy region

Definition 2.3.1. *A fuzzy region A is represented as $A = \{(x, y), \mu_A(x, y)\}$ where $(x, y) \in \mathbb{R}^2$ and $\mu_A : \mathbb{R}^2 \to [0, 1]$.*

Definition 2.3.2. *A simple fuzzy region is a non-empty subset of an fts which is a double connected regular closed set whose core is the interior of a simple closed region, the i-closure is a non-empty double connected regular open set, the interior of the b-closure is a non-empty double-connected regular open set, the c-boundary is a non-empty double-connected closed set and the outer is a non-empty double-connected open set. Various subsets of a simple region are shown in Figure 1.*

2.4 Topological relations between two simple regions

As in [4] and [6] for crisp topological spaces, for description of topological relations between 2 fuzzy sets in an fts, A and B, one can use the *9-intersection matrix* or the 4×4 *intersection matrix* [15, 16].

$$I_9 = \begin{pmatrix} A^{\oplus} \cap B^{\oplus} & A^{\oplus} \cap \ell B & A^{\oplus} \cap B^{=} \\ \ell A \cap B^{\oplus} & \ell A \cap \ell B & \ell A \cap B^{=} \\ A^{=} \cap B^{\oplus} & A^{=} \cap \ell B & A^{=} \cap B^{=} \end{pmatrix}, \tag{2.1}$$

$$I_{4\times 4} = \begin{pmatrix} A^{\oplus} \cap B^{\oplus} & A^{\oplus} \cap B^{\perp} & A^{\oplus} \cap \partial^c B & A^{\oplus} \cap B^{=} \\ A^{\perp} \cap B^{\oplus} & A^{\perp} \cap B^{\perp} & A^{\perp} \cap \partial^c B & A^{\perp} \cap B^{=} \\ \partial^c A \cap B^{\oplus} & \partial^c A \cap B^{\perp} & \partial^c A \cap \partial^c B & \partial^c A \cap B^{=} \\ A^{=} \cap B^{\oplus} & A^{=} \cap B^{\perp} & A^{=} \cap \partial^c B & A^{=} \cap B^{=} \end{pmatrix}. \tag{2.2}$$

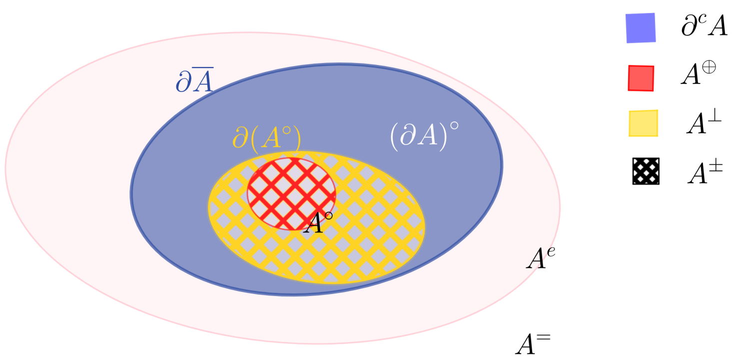


Figure 2.1: A pictorial example of a simple region

Of all the $2^9 = 512$ relations possible between 2 fuzzy sets from the 9-intersection matrix, only 44 can happen for a simple fuzzy region. With the same restrictions imposed, of the $2^{16} = 65536$ relations derived from the 4×4 intersection matrix, only 152 remain feasible for a simple fuzzy region.

Since processing spatial information often involves dealing with characteristics which are inexact in a way, dealing with non-exact objects has quite some importance in geographical information systems (GIS). The difficulty in extending the methods of crisp topology to a fuzzy domain lies in choosing the most apt and sensible generalization among the large number of possible approaches.

2.5 Entropy

We define $\mathcal{F}(X)$ as the class of all fuzzy sets on X and $\mathcal{P}(X)$ be the class of all crisp sets on X. Liu Xuechung in [19] defined this set

Definition 2.5.1. $\left[\frac{1}{2}\right]_X \in \mathcal{F}(X) \ni \mu_{\left[\frac{1}{2}\right]_X}(x) = \frac{1}{2}\ \forall\ x \in X.$

Let us define $\mathcal{F} \subseteq \mathcal{F}(X) \ni$

1. $\mathcal{P}(X) \subset \mathcal{F}$,
2. $\left[\frac{1}{2}\right]_X \in \mathcal{F}$,
3. $A, B \in \mathcal{F} \implies A \cup B \in \mathcal{F}, A^C \in \mathcal{F}$.

Definition 2.5.2. *A real function* $e : \mathcal{F} \to \mathbb{R}^+$ *s called an entropy on* $\mathcal{F}$ *if* e *has the following properties*

1. $e(D) = 0\ \forall\ D \in \mathcal{P}(x)$;
2. $e\left(\left[\frac{1}{2}\right]_X\right) = \max_{A \in \mathcal{F}} e(A)$;
3. $\forall\ A, B \in \mathcal{F}$, *and either* $\mu_B(x) \geq \mu_A(x) \geq \frac{1}{2}$ *or* $\mu_B(x) \leq \mu_A(x) \leq \frac{1}{2}$, *then* $e(A) \geq e(B)$;
4. $e\left(A^C\right) = e(A)\ \forall\ A \in \mathcal{F}$.

Example. Let $X = \{x_1, x_2, \ldots, x_n\}$. For any $A \in \mathcal{F}(X)$, $\hat{A} \in \mathcal{P}(X) \ \ni$

$$\mu_{\hat{A}}(x) = \begin{cases} 1 & \text{when } \mu_A(x) > \frac{1}{2} \\ 0 & \text{when } \mu_A(x) \leq \frac{1}{2} \end{cases}$$

and $e'(A) = \left(\sum_{i=1}^{n} \left|\mu_A(x_i) - \mu_{\hat{A}}(x_i)\right|^{\omega}\right)^{\frac{1}{\omega}}\ \forall\ A \in \mathcal{F}(X)$. Then e' is an entropy on $\mathcal{F}(X)$, where $\omega \geq 1$.

Definition 2.5.3. *Further, if an entropy* e *on* $\mathcal{F}$ *satisfies* $e\left(\left[\frac{1}{2}\right]_X\right) = 1$, *then* e *is called a* normal entropy *on* $\mathcal{F}$.

Result 2.5.4. *Let* e *be an entropy pf* $\mathcal{F}$, *then* $\hat{e}$ *given by* $\hat{e}(A) = \dfrac{e(A)}{e\left(\left[\frac{1}{2}\right]_X\right)}\ \forall\ A \in \mathcal{F}$ *is a normal entropy on* $\mathcal{F}$.

$\left[\frac{1}{2}\right]_X$ is used in deriving core results regarding entropy. In the next chapter, a similar subset of a fuzzy set has been defined and its properties have been studied in detail with sole attention. Before that, a certain aspect of another work is showed in brief, that uses similar sets for computation.

2.6 Distribution of SARS among people in a community

In [14], the set $A_{1/2}$ has been defined such that

$$\mu_{A_{1/2}}(x) = \begin{cases} \mu_A(x) & \text{if } \mu_A(x) > \frac{1}{2}; \\ 0 & \textit{otherwise}. \end{cases}$$

Using concepts of quasi-difference and quasi-coincidents, a seven-tupled topological relation was defined for two fuzzy objects, to be used in GIS to quantify the effect of one fuzzy entity on another.

To detect the distribution of SARS in a community by tracing the path of an affected person, investigating the effect of that person as well as an infected region on the community, five topologically invariant components:

1. $(A \cap B)_{\mu_A + \mu_B \leq 1}$,
2. $\bigcap (A \cap B)^{\mu_A \text{ or } \mu_B \leq 0.5}_{\mu_A + \mu_B > 1}$
3. $\bigcap \left(A \cap B^C\right)^{\mu_A \text{ and } \mu_B > 0.5}_{\mu_A + \mu_B > 1}$,
4. $\bigcap \left(A^C \cap B\right)^{\mu_A \text{ and } \mu_B > 0.5}_{\mu_A + \mu_B > 1}$,
5. $\{x \in X | \mu_A(x) = \mu_B(x)\}$

were used, using the idea of a set exclusively containing those points whose membership is 1/2. Study of such a set which can be beneficial to the above procedures, has been conducted in the next chapter.

Chapter 3

Coast

3.1 Defining Coast

We next define something we can visualize as a line that demarcates the elements more inclined towards a fuzzy set than its complement. Motivations behind this explicit study is the use of related concepts in [19, Example 2.2] and [14] for exploring ideas like entropy and applications like spread SARS respectively, among others.

Let A be a fuzzy set. The *Coast* of A is the set of those elements whose degrees of membership in A are exactly $\frac{1}{2}$. Alternatively, seeing the Coast as a subset of $\overline{A}$ makes more sense because there might be elements in $\overline{A}$ for which the membership function in A is $\frac{1}{2}$. As a third option, we can define the Coast as a subset of the boundary (B2).

Definition 3.1.1. *1. The first Coast $A_L^{(1)}$ of a fuzzy set A is defined by the membership function*

$$\mu_{A_L^{(1)}}(x) = \begin{cases} \mu_A(x) & \text{if } \mu_A(x) = \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}.$$

2. The second Coast $A_L^{(2)}$ of a fuzzy set A is defined by the membership function

$$\mu_{A_L^{(2)}}(x) = \begin{cases} \mu_{\overline{A}}(x) & \text{if } \mu_{\overline{A}}(x) = \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}.$$

3. The third Coast $A_L^{(3)}$ of a fuzzy set A is defined by the membership function

$$\mu_{A_L^{(3)}}(x) = \begin{cases} \mu_{\overline{A}\cap\overline{A^C}}(x) & \text{if } \mu_{\overline{A}\cap\overline{A^C}}(x) = \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}.$$

We can also imagine this third definition as a boundary line *that makes a line demarcation out of the fuzzy boundary.*

Note that $A_L^{(1)}$ is the fuzzy set consisting of elements which have same membership function values in A and A^C. We define $L = \operatorname{supp}\left(A_L^{(1)}\right)$ as its crisp version, i.e., $L = \operatorname{supp}\left(A_L^{(1)}\right) = \left\{x \in X \ : \ \mu_A(x) = \frac{1}{2}\right\}$.

Definition 3.1.2. *The* enclosure *of a Coast is the interior of $\left\{x \in X \ : \ \mu_A(x) \leq \frac{1}{2}\right\}$, i.e., $In(A_L) = \left\{B \subseteq X \ : \ h(B) \leq \frac{1}{2}\right\}$.*

3.2 Properties of Coast

Result 3.2.1. *If A is closed, then $A_L^{(1)} = A_L^{(2)}$.*

Proof. If A is closed, then $A = \overline{A}$ so exactly those elements which have membership $\frac{1}{2}$ in A shall have membership $\frac{1}{2}$ in $\overline{A}$.

Result 3.2.2. *If A is closed, $A_L^{(1)} \subset A_L^{(3)}$.*

Proof. Let $\mu_A(x) = \frac{1}{2} \implies \mu_{\overline{A}}(x) = \mu_A(x) = \frac{1}{2}$ and $\mu_{\overline{A^C}}(x) \geq \frac{1}{2} \implies \mu_{\overline{A}} \wedge \mu_{\overline{A^C}}(x) = \frac{1}{2}$. □

Theorem 3.2.3. *If A is clopen, $A_L^{(1)} = A_L^{(3)}$.*

Proof. Let $x \in A_L^{(3)} \implies \mu_{\overline{A} \wedge \overline{A^C}}(x) = \frac{1}{2}$. If $\mu_{\overline{A^C}}(x) = \frac{1}{2}$ then $\mu_{A^C}(x) = \frac{1}{2}$ [$\because A$ is open $\implies A^C$ is closed $\implies \mu_{A(x)} = \frac{1}{2}$. If $\mu_{\overline{A}}(x) = \frac{1}{2}$, then $\mu_A(x) = \frac{1}{2}$ [$\because A$ is closed]. $\therefore A_L^{(1)} \supset A_L^{(3)}$. The rest follows from Result 3.2.2. □

Theorem 3.2.4. $A_L^{(2)} < A_L^{(3)}$. *In particular, if A is clopen, $A_L^{(2)} \subset A_L^{(3)}$.*

Proof. Follows from the fact that $\mu_{\overline{A}}(x) = \frac{1}{2} \implies \mu_A(x) \leq \frac{1}{2} \implies \mu_{A^C}(x) \geq \frac{1}{2}$. □

Theorem 3.2.5. $A_L^{(1)}$ *is a topological invariant.*

Proof. Let us consider an I-fuzzy homemorphism $f^{\rightarrow} : \left(I^X, \delta\right) \rightarrow \left(I^Y, \mu\right)$. Then, $\forall\, y \in Y\ \exists$ unique $x_0 \in X \ni f(x_0) = y$. Thus, $f^{\rightarrow}(\mu_A)(y) = \vee\{\mu_A(x) : x \in X \ni f(x) = y\} = \mu_A(x_0)$. Now,

$$\begin{aligned} f^{\rightarrow}\left(\mu_{A_L^{(1)}}\right)(y) &= \begin{cases} f^{\rightarrow}\left(\mu_{A_L^{(1)}}\right)(x_0) & \text{if } \mu_A(x_0) = \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \mu_{A_L^{(1)}}(x_0) & \text{if } \mu_A(x_0) = \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \mu_A(x_0) & \text{if } \mu_A(x_0) = \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}. \end{aligned}$$

Again,

$$\begin{aligned} \left(f^{\rightarrow}(\mu_A)\right)_L^{(1)}(y) &= \begin{cases} f^{\rightarrow}(\mu_A)(y) & \text{if } f^{\rightarrow}(\mu_A)(y) = \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \mu_A(x_0) & \text{if } \mu_A(x_0) = \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}. \end{aligned}$$

$\therefore f^{\rightarrow}\left(\mu_{A_L}^{(1)}\right)(y) = \left(f^{\rightarrow}(\mu_A)\right)_L^{(1)}(y)\ \forall\, y \in Y$, i.e., $\mu_{A_L^{(1)}}$ is an invariant of homeomorphism. □

Theorem 3.2.6. $A_L^{(1)} \subseteq A_L^{(3)}$.

Proof. Let $x \in A_L^{(1)}$. $\therefore \mu_A(x) = \frac{1}{2} \implies \mu_{\overline{A}}(x) = \frac{1}{2} = \varepsilon$ for some $\varepsilon \geq 0$. Again, $\mu_{A^C}(x) = 1 - A(x) = \frac{1}{2} \implies \mu_{\overline{A^C}}(x) = \frac{1}{2} + \varepsilon'$ [$\because \mu_A \leq \mu_{\overline{A}}$]. But then one of A and A^C must be closed. Let $B \in \{A, A^C\}$ is closed. Then $\overline{B}(x) = B(x) = \frac{1}{2}$. Then $\left(\mu_{\overline{A}} \wedge \mu_{\overline{A^C}}\right)(x) = \frac{1}{2} \implies x \in \overline{A} \cap \overline{A^C}$. $\therefore A_L^{(1)} \subseteq A_L^{(3)}$. □

3.3 Examples

1. Let us suppose A be a fuzzy set in X where

$$\mu_A = \begin{cases} 0 & \text{if } |x| > 3, \\ \varepsilon & \text{if } |x| = 3, \\ ||x| - 3| & \text{if } 2 < |x| < 3, \\ 1 & \text{if } |x| \leq 2. \end{cases}$$

Then A is closed in $\mathbb{R}$ as it is upper semicontinuous from $\mathbb{R}$ to I. Then,

$$\mu_{A^\circ} = \begin{cases} 0 & \text{if } |x| > 3, \\ 0 & \text{if } |x| = 3, \\ ||x| - 3| & \text{if } 2 < |x| < 3, \\ 1 - \varepsilon_1 & \text{if } |x| = 2, \\ 1 & \text{if } |x| < 2. \end{cases}$$

Here, $L = \{\pm 2.5\}$ and $\mu_{A_L^{(1)}} = \left\{ \left(2.5, \frac{1}{2}\right), \left(-2.5, \frac{1}{2}\right) \right\}$.

3.4 Results about Coast similar to Warren's properties

It should be repeated again that our definition of the Coast is not intended to serve as a definition of boundary. On the very contrary, is it applicable only in those cases where we find it difficult to deal with more formal definitions of fuzzy boundary but still would like our sets to retain their innate sense of fuzziness. It is only in those cases that we would like a "line" between a set and its complement, a line that marks the bounds of the set. It gives us a somewhat crisp boundary in an otherwise fuzzy environment. But its application lies in an entirely different environment than where fuzzy boundary is applied.

Richard Warren postulated some conditions a reasonably good definition for fuzzy boundary (∂A) must satisfy [the definitions that satisfy a particular property are written in brackets]:

(W1) ∂A is closed. [(B1), (B2), (B3), (B4)]

(W2) The closure is the supremum of the interior and the boundary, i.e., $\overline{A} = A^\circ \vee \partial A$. [(B1), (B2), (B3), (B4)]

(W3) This fuzzy definition of boudary becomes the usual topological boundary when the sets concerned are crisp. [(B1), (B2), (B3), (B4)]

(W4) The boundary operator is an equivalent way of defining a fuzzy topology. [(B1), (B2), (B3), (B4)]

(W5) The boundary of a fuzzy set is identical to the boundary of the complement of the set. [(B2), (B4)]

(W6) If a fuzzy set is closed (or open), then the interior of the boundary is empty. [(B3)]

(W7) If a fuzzy set is both open and closed, then the boundary is empty.

A Coast has nothing to do with these properties, and indeed, satisfying them is not a criteria. We have in this section formulated some properties the coast satistfies. These properties are comparabe with Warren's properties and hence might give a clearer idea of the comparison and contrast between fuzzy boundary and Coast.

(W1′) Unlike the conventional boundaries, Coast is not closed.

(W2′) $\overline{A} > A^\circ \vee A_L^{(1)}$, $\overline{A} > A^\circ \vee A_L^{(2)}$ however it is not necessarily the supremum as stated in the condition.

(W5′) holds for $A_L^{(1)}$ and $A_L^{(3)}$ but not necessarily for $A_L^{(2)}$.

(W6′) If the interior of the Coast in $\mathbb{R}^2$ is empty, then the Coast is a line or a set of scattered points. Else, it is two-dimensional.

Proof.

(W2′) We know, $\overline{A} \supset A^\circ$. Also $\mu_{\overline{A}}(x) \geq \mu_A(x) \implies \mu_{\overline{A}}(x) \geq \frac{1}{2}$ when $\mu_{A(x)} \geq \frac{1}{2}$. So, $\overline{A} \supset A^\circ \cup A_L^{(1)}$. The other part follows the same way.

(W5′) This holds since $\mu_A(x) = \frac{1}{2} \implies \mu_{A^C}(x) = \frac{1}{2}$.
For $A_L^{(2)}$ this is true only if $\mu_A(x) = \mu_{\overline{A}}(x)$, i.e., if A is closed.
For $A_L^{(3)}$, we can see that $\mu_{\overline{A}} \wedge \mu_{\overline{A^C}} = \frac{1}{2} \Leftrightarrow \mu_{\overline{(A^C)^C}} \wedge \mu_{\overline{A^C}} = \frac{1}{2}$. So the Coasts for A and A^C are same.

(W6′) It is easy to see that every two-dimensional area has a non-empty interior. So, if the interior is empty, the Coast must be a line or a scattered set of points. Figures 8.4, 8.5 and 8.6 shows that the Coast can indeed be two-dimensional.

3.5 Coast forms a subset of the boundary

By outward monotonicity, we mean that whenever the central point of the region (with membership 1) is joined with a point on the boundary by a straight line, with distance from the center the membership value monotonically decreases (or remains same). Thus the points with membership $\frac{1}{2}$ should lie together on each such line. Continuity over $\mathbb{R}^2$ implies that those points shall form a connected set.

If the membership value is strictly decreasing on each outward line, then each line can have at most 1 point with membership $\frac{1}{2}$. However, continuity implies that running from 1 to 0, the line shall have memberships each value in between. So, A_L is a one-dimensional loop. Thus,

Theorem 3.5.1. *In a simple closed region with outwardly monotone and continuous membership function, A_L is a closed loop. If it is strictly decreasing outward then A_L is a 1-dimensional loop.*

Chapter 4

Use of Coast

4.1 As a 'crisper' subset of the boundary

Broadly, any point between absolute membership 1 and 0 is a boundary point for a fuzzy set. However, a more precise categorization of boundary is necessary. For image enhancement we sometimes need a clearer view of the set and its exterior, and Coast solves this purpose to some extent.

Result 4.1.1. $0 \leq \mu_{\partial A} \leq \frac{1}{2}$ *for any fuzzy set A.*

The membership of a point in the boundary is at most half. The more we go both ways to the interior or exterior of the set, membership in the boundary decreases. The definition of Coast has been conceived such that it considers as boundary, in a very crisp way, only those points which have the maximum possible membership in other definitions of boundary, making $A_L^{(3)}$ a proper subset of our usual definition of ∂A and other A_L's alternatives in appropriate situations.

This gives a thinner boundary, demarcating the set and its exterior more prominently.

Example. Where the exact shoreline varies due to daily high and low tides and lunar cycles, the Coast can be used to draw a precise outline map. We can paint the interior as inland and the exterior blue for the sea. For a gross idea or for a political map this will be more suited.

The sand beach is usually the boundary between land and water and being a part of each during each time of the day. Now, if A denotes underwater place, $\mu_A(x) = 0$ implies the place x is inland. Similarly, $\mu_A(x) = 1$ for the sea.

One likely option is to mark the exact water-lines during the high-tides and low-tides everyday over a lunar cycle and take their average mid-point on map to mark the place with membership m_A as $\frac{1}{2}$. But this approach has its problems that (i) the slope of the land is not reflected in the map; (ii) land undulations or nature of rocks and soils can affect tide water level, and hence these marked points may not exactly reflect 'the state of being halfway inland'.

As a better alternative, we can set $\mu_A(x) = \frac{1}{2}$ is the point x remains dry for exactly 12 hours a day in all, or on an average $\frac{1}{2}$ of the lunar cycle. For political or jurisdictory maps (which indeed is needed very much in river-delta areas, where calculation of the areas of islands is necessary to estimate the amount of land under each police station and fix patrolling duties) or for drawing one-pixel wide coastal line in maps, these points with μ_A as $\frac{1}{2}$ can be used. Then we consider the border as A_L.

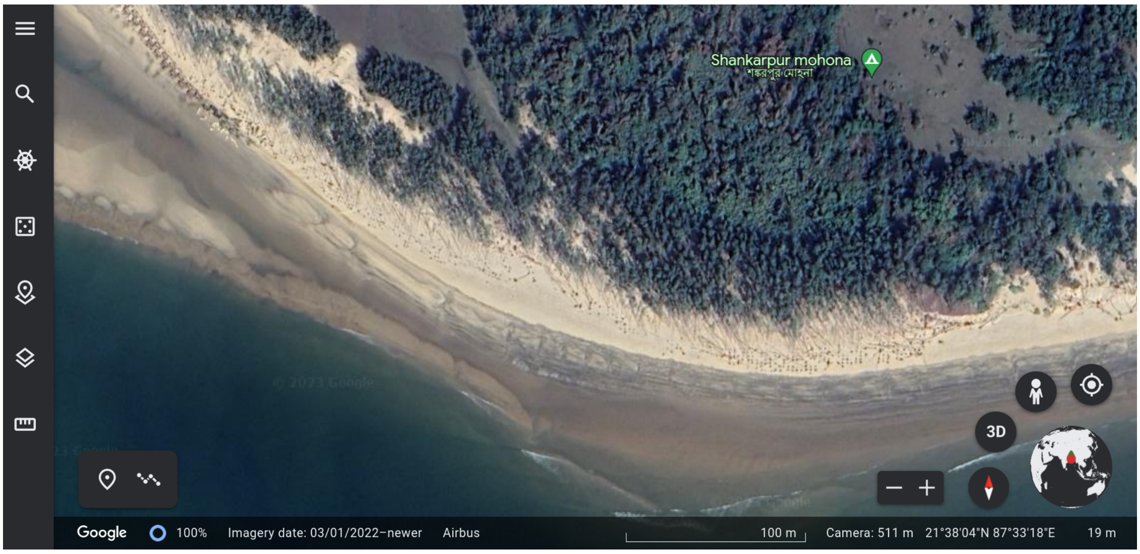


Figure 4.1: Sankarpur Mohana, West Bengal from Google Earth. Here Google Earth uses some technique to make one coast line out of several time series images.

An example is given from NASA Worldview where they use coastal linings. The off-white area by the coast is the fuzzy sandbeach in Fig 8.2 while the black demarcation line is drawn through that in Fig 8.3.

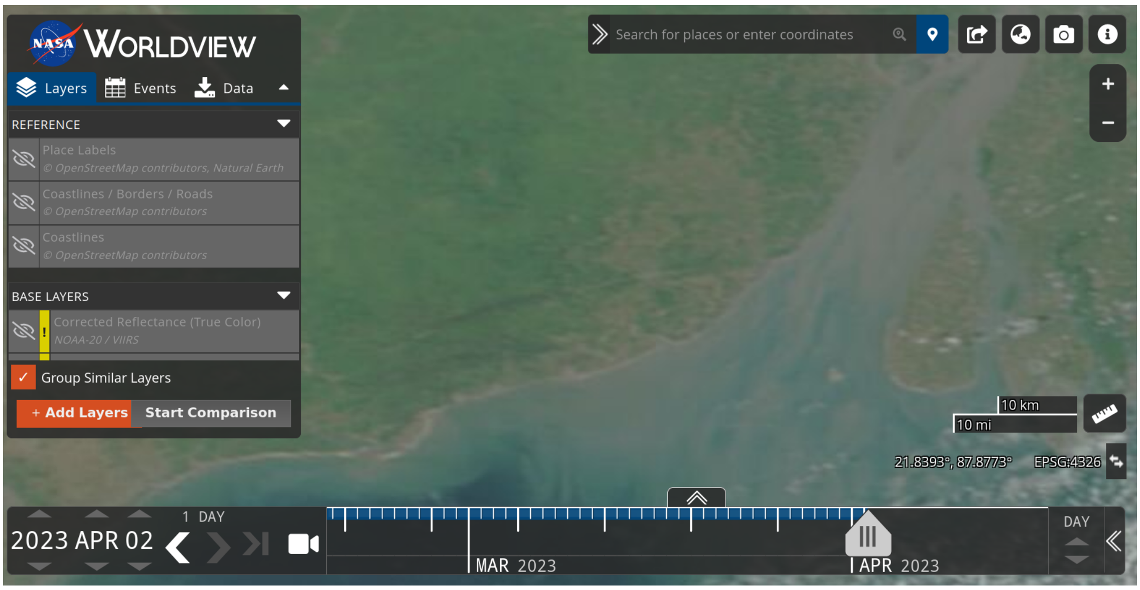


Figure 4.2: Western part of Bhagirathi-Hoogly delta

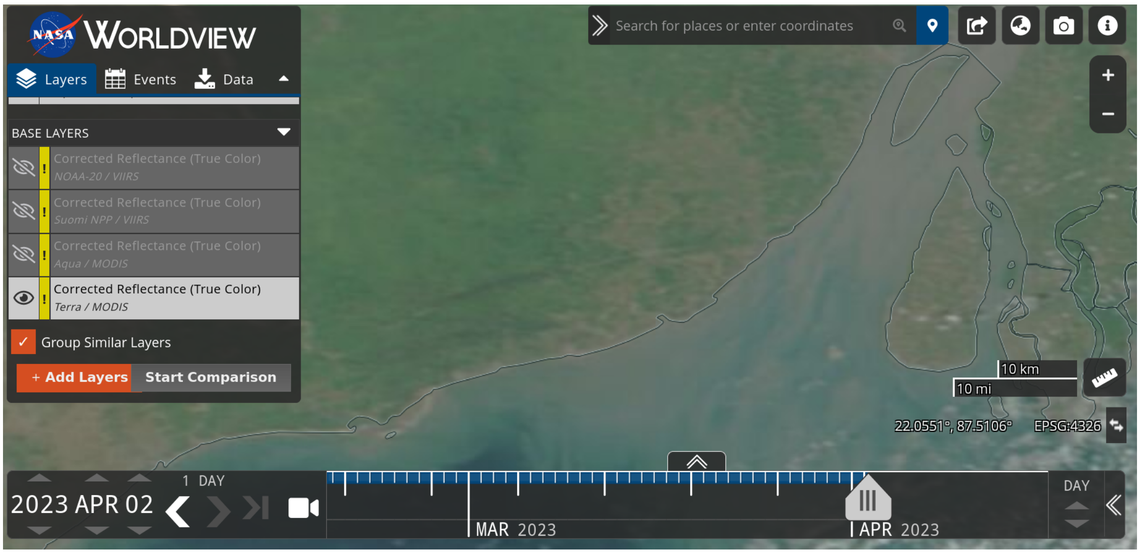


Figure 4.3: Western part of Bhagirathi-Hoogly delta, coast outlined

4.2 Calculation of Coast in the above-posed situation

We take $2n$ number of aerial photographs of the beach in question within the same frame over a lunar cycle, captured at regular time intervals. Superimposing the images, we get $2n$ water-lines stretched almost all across the beach width. We mark black the strip between the n and $(n+1)$th lines as the *Coast*. Of course, this is not exactly the Coast as defined, but the best possible approximation from our pictorial data. No better conclusion from the given assumed data is possible. Each point within this strip remains underwater in exactly half of the pictures.

As n increases, we get a finer strip that approaches the exact fuzzy line $\left\{x \mid \mu_A(x) = \frac{1}{2}\right\}$ as $n \to \infty$. It might be noted as well, that a fuzzy line may or may not be a line. If the slope at one point is 0 and its membership is $\frac{1}{2}$, then the fuzzy line is two-dimensional there. Figures 8.4, 8.5 and 8.6 show such a case where the whole strip BC has μ_A $\frac{1}{2}$.

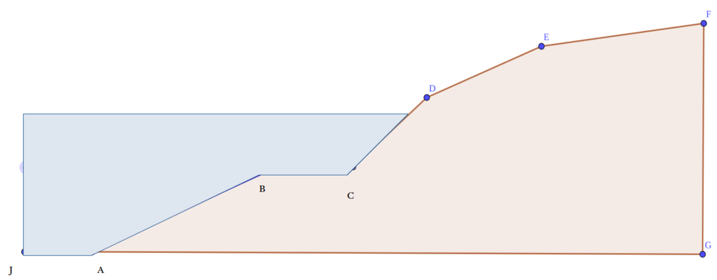


Figure 4.4: The whole strip BC has $\mu_A = \frac{1}{2}$: when the water level is above BC

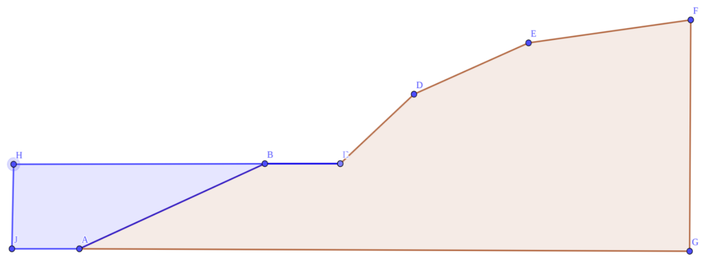


Figure 4.5: The whole strip BC has $\mu_A = \frac{1}{2}$: when the water level is at BC

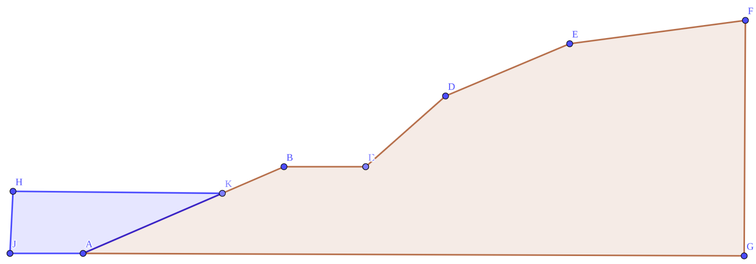
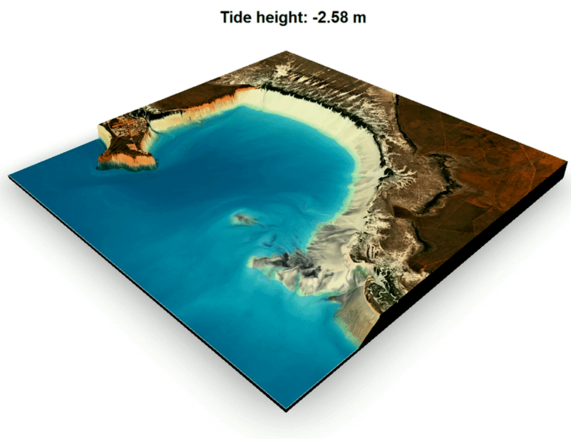


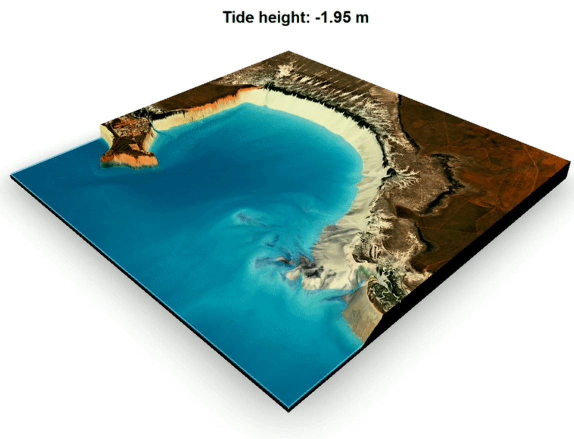
Figure 4.6: The whole strip BC has $\mu_A = \frac{1}{2}$: when the water level is below BC

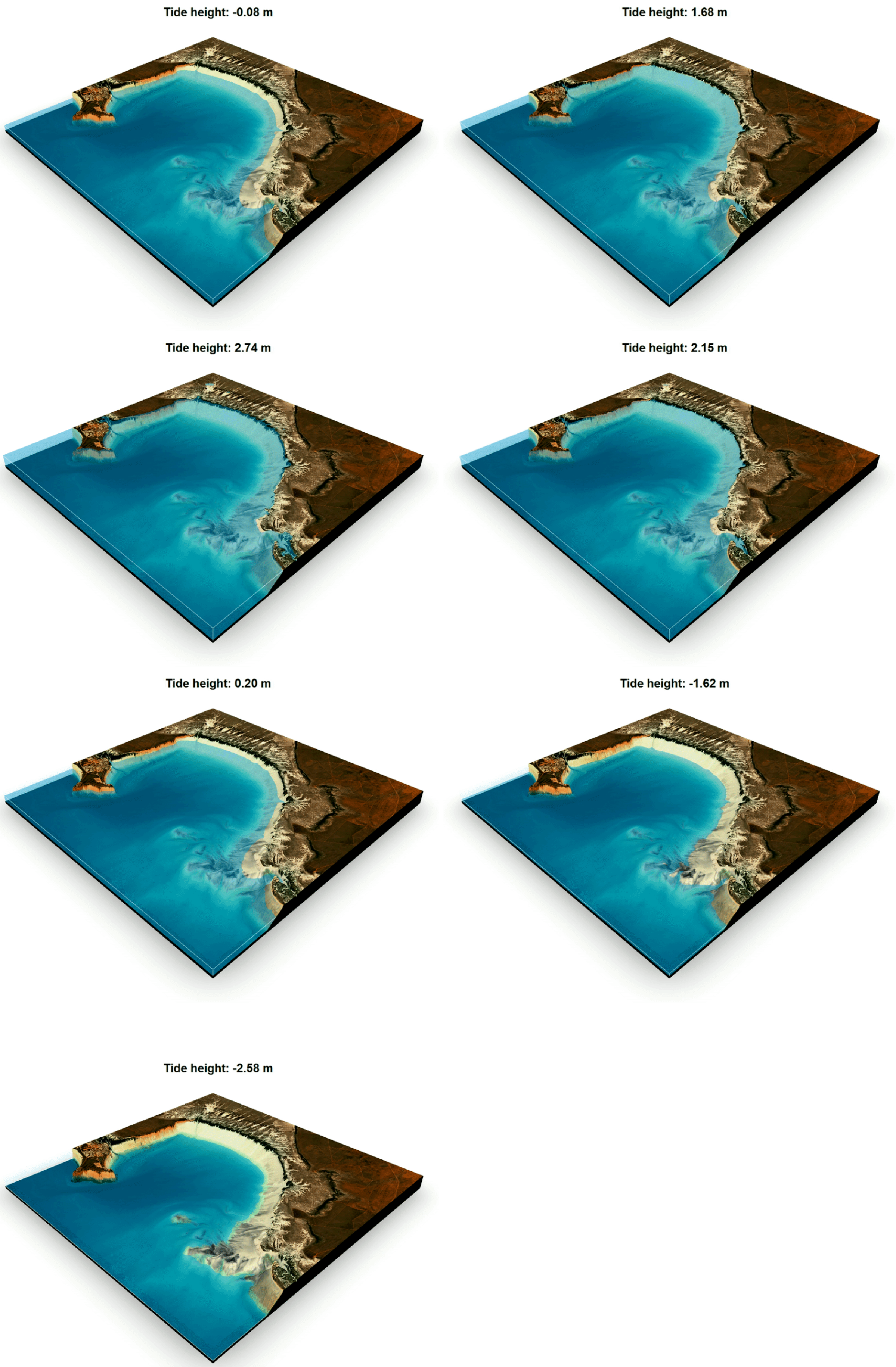
Also, Theorem 3.5.1 might or might not be applicable. The slope at some point might be infinity, and hence the graph discontinuous. Take for example rocky cliff sea-shores which naturally give us Coasts predefined.

4.3 An example

The data used has been taken from `https://earthobservatory.nasa.gov/images/145237/mapping-the-land-between-the-tides`. A 3D Landsat model of Roebuck coast of Australia is shown below, each photograph of time 3 hours apart in a 24-hour cycle.






Tide height: -0.08 m
Tide height: 1.68 m
Tide height: 2.74 m
Tide height: 2.15 m
Tide height: 0.20 m
Tide height: -1.62 m
Tide height: -2.58 m

Superimposed, the images give a Coast somewhere through the middle of the beach.

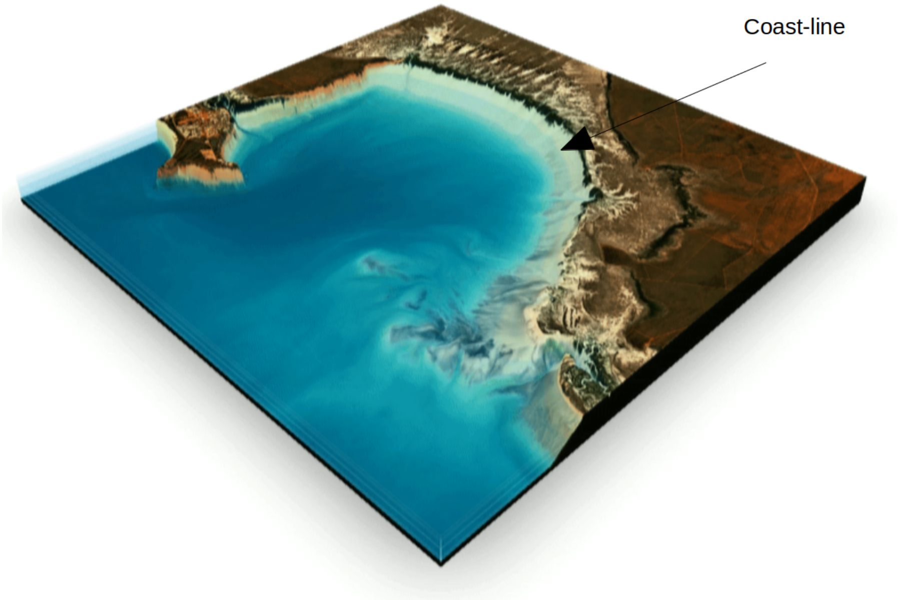


Figure 4.7: Coast given by superimposing 8 frames a day

Below, the Coast given by roughly the 29th, 30th, 87th and 88th frames is shown, from a 116-frame data of the same shore over a diurnal tide cycle.

Figure 4.8: Coast given by superimposing 116 frames a day

Chapter 5

Conclusions

From the use of related concepts in [19, Example 2.2] and [14] we have essentially formulated here 3 definitions for the coast of a fuzzy set. There might be situations in which each shall prove useful; broadly speaking, the coast consists of those points in the boundary whose membership either in the set or its closure or in the boundary is $\frac{1}{2}$. We have further shown how these three coastlines interact among themselves, as well as with the standard definitions of fuzzy boundary. Examples have been illustrated to show some situations in which dealing with the coast is more appropriate than the fuzzy boundary. Demarcation between land and water for judicial maps on a concrete basis is feasible with this idea. Similar procedures might be followed for other geographical characteristics like forest cover (trees and clearing) or soil types.

It might be worth further study how the various boundaries interact with the Coast, and what benefits one might derive in other problems by adopting the Coast as demarcation between complementary sets. Further techniques may be employed in cases where the coast in a map is two-dimensional, i.e., has a non-empty interior. It might need to be further reduced to a one-dimensional subset for the sake of border clarity.

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