Taylor Series for Real Numbers

We know that if $a \in \mathbb{R}$, by denseness, $\exists$ a sequence $\{a_n\}_{n \in \mathbb{N}}$ of real nos. such that $a_n \to a$ as $n \to \infty$. For a fixed value of x, a function $f : \mathbb{R} \to \mathbb{R}$ gives a real no. as output, and so, there must be a sequence (depending on x, i.e., entries being functions of x) that converges to x. By Weierstrass' approximation theorem, we understand that every function can be locally approximated up to any required accuracy by some polynomial function. So, if we want a sequence of functions of x to converge to $f(x)$ at or around a particular point x, what's better than to consider this sequence of partial sums $\left\{ \sum_{i=0}^{n} a_i \cdot x^i \right\}_{n \in \mathbb{N}}$ which should converge to $f(x)$ for some a_i - values. Now, if $\sum_{i=0}^{n} a_i \cdot x^i$ converges as $n \to \infty$, then its limit is equal to $\sum_{i=0}^{\infty} a_i \cdot x^i$, which must be equal to $f(x)$ at x. If, $f(x)$ is indeed a polynomial of some finite degree m, then $a_i = 0$ for $i > m$.

Earlier, while computing transcendetal functions, this was a popular method : finding the polynomial and computing it up to any required no. of terms. But for that one needs to find the coefficients a_i $(i \in \mathbb{N} \cup \{0\})$ and have the polynomial with all the required terms at one's disposal. Let us see how to do that :

Let $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots$ We are trying to find the polynomial when $x = 0$, to begin with. Assume f is continuous, and each derivative of each derivative of f is again differentiable. Then,

$$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots \Rightarrow f'(0) = a_1$$

$$f''(x) = 2a_2 + 3.2\, a_3 x + 4 \cdot 3 \cdot a_4 x^2 + \cdots \Rightarrow f''(0) = 2a_2$$

$$f'''(x) = 3.2\, a_3 + 4.3.2 \cdot a_4 x + \cdots \Rightarrow f'''(0) = 3.2.\, a_3, \quad f''''(0) = 4.3.2.\, a_4$$

In this way, $\underset{\substack{\uparrow \\ n^{\text{th}} \text{ derivative of } f}}{f^{(n)}}(0) = n!\, a_n \Rightarrow a_n = \dfrac{f^{(n)}(0)}{n!}$.

Thus we know the entire polynomial now.

<u>Example 1.</u> $f(x) = e^x \Rightarrow f^{(n)}(x) = e^x \Rightarrow f^{(n)}(0) = 1 \Rightarrow a_n = \dfrac{1}{n!}$

$$\therefore e^x = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \cdots$$

EXAMPLE 2. Try finding in this way, without using De Moivre's formula, polynomial approximations of $\sin x$ and $\cos x$.

EXAMPLE 3. The binomial expansion

$$f(x) = (1+x)^a = 1 + \frac{a}{1}x + \frac{a(a-1)}{2!}x^2 + \frac{a(a-1)(a-2)}{3!}x^3 + \dots$$

TAYLOR SERIES AT ANOTHER BASE POINT

If we want the Taylor series at another base point, it is easier to define a translation, taking that base point to 0, find the Taylor series there, and shifting back to the old coordinates:

$$f(x) = f(b) + f'(b)(x-b) + \frac{f''(b)}{2}(x-b)^2 + \frac{f^3(b)}{3!}(x-b)^3 + \dots \qquad \text{at } x = b$$

For example, if you want the Taylor series for $\sqrt{x}$, you can't find it at $x=0$, because f is not differentiable at 0 so $f'(0)$ is not defined. At $x=1$,

$$x^{1/2} = 1 + \frac{1}{2}(x-1) + \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)}{2!}(x-1)^2 + \dots$$

If you want to differentiate a function but don't know what its' derivative will be, you can find its Taylor series and differentiate that. Try finding the derivative of

$\sinh x = \dfrac{e^x - e^{-x}}{2}$ at $x = 2$, and try integrating the same function between bounds $x=1$ and $x=2$.

[based on discussion in the end of MAL101 Tutorial class on June 13th 2025, in which unfortunately only two students participated]