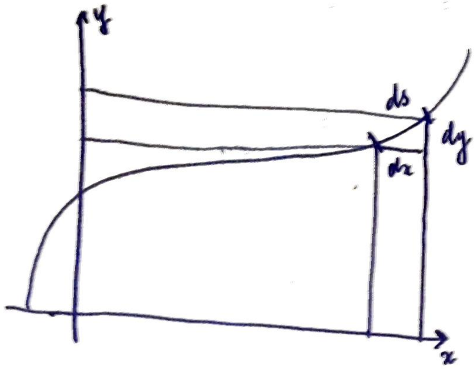
$$y = \frac{2}{3}(x^2+1)^{\frac{3}{2}} \qquad (0,2)$$

ARC LENGTH



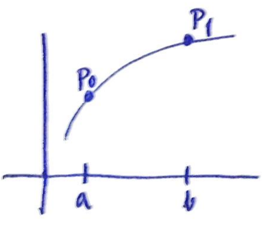
$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

Integrating with respect to ds finds the length of a curve between two points. To find the length of a curve between P_0 and P_1, evaluate:

$$\int_{P_0}^{P_1} ds$$

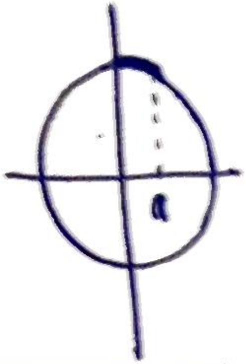
We want to integrate with respect to x, not s, so we do the same algebra as above ~~and~~ to find ds in terms of dx.

$$\frac{(ds)^2}{(dx)^2} = \frac{(dx)^2}{(dx)^2} + \frac{(dy)^2}{(dx)^2} = 1 + \left(\frac{dy}{dx}\right)^2.$$



$$\therefore \int_{P_0}^{P_1} ds = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

Ex The circle $x^2+y^2=1$ $\qquad y=\sqrt{1-x^2}$



$$\frac{dy}{dx} = \frac{-2x}{\sqrt{1-x^2}}\left(\frac{1}{2}\right) = \frac{-x}{\sqrt{1-x^2}}$$

$$ds = \sqrt{1+\left(\frac{-x}{\sqrt{1-x^2}}\right)^2}\;dx$$

$$= \sqrt{1+\left(\frac{-x}{\sqrt{1-x^2}}\right)^2}\,dx = \sqrt{1+\frac{x^2}{1-x^2}}\,dx = \sqrt{\frac{1-x^2+x^2}{1-x^2}}\;dx = \frac{1}{\sqrt{1-x^2}}\,dx$$

$$ds = \sqrt{\frac{1}{1-x^2}}\;dx$$

$$s = \int_0^a \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x\Big|_0^a = \sin^{-1}a - \sin^{-1}0 = \sin^{-1}a$$

$$\Rightarrow \sin s = a$$

Parametric equations

Ex.

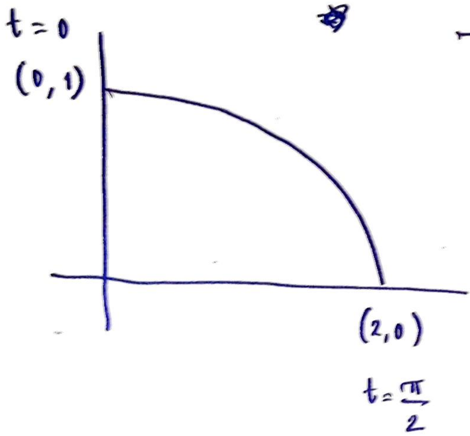
~~$x = a\cos t$~~

(Ellipse) ~~$y = a\sin t$~~

$x = 2\sin t \,;\; y = \cos t$ | ~~$dx = -a\sin t\, dt$~~

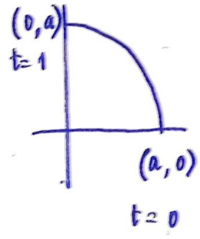
$$\frac{x^2}{4} + y^2 = 1$$

~~$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{}$~~



Ex. $x = a\cos t$ | For $0 \le t \le \frac{\pi}{2}$, a quarter circle is traced counterclockwise.

$y = a\sin t$



$dx = -a\sin t\, dt, \quad dy = a\cos t\, dt$

$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{(-a\sin t\, dt)^2 + (a\cos t\, dt)^2}$$

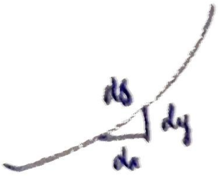
$$= \sqrt{(a\sin t)^2 + (a\cos t)^2}\, dt = a\, dt.$$

error problems
proof of GP series

Ex. Consider this parametric eqn

$x = t^2, \; y = t^3 \quad 0 \le t \le 1$

$\longrightarrow y = x^{2/3} \quad 0 \le x \le 1$



$$(ds)^2 = (dx)^2 + (dy)^2$$

$$= \underbrace{(2t\,dt)^2}_{(dx)^2} + \underbrace{(3t^2\,dt)^2}_{(dy)^2} = (4t^2 + 9t^4)(dt)^2$$

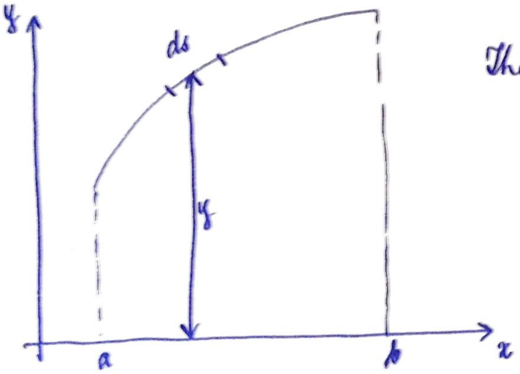
$$\text{Length} = \int_{t=0}^{t=1} ds = \int_0^1 \sqrt{4t^2 + 9t^4}\,dt = \int_0^1 t\sqrt{4+9t^2}\,dt$$

$$= \frac{(4+9t^2)^{3/2}}{27}\Bigg|_0^1 = \frac{1}{27}\left(13^{3/2} - 4^{3/2}\right)$$

Surface area (surfaces of revolution)

ds is revolved around a distance $2\pi y$.

The surface area of the thin strip ds is $2\pi y\, ds$

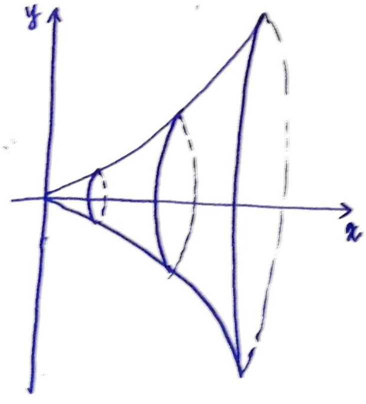


$$\text{Area} = \int 2\pi y\, ds = \int_0^1 2\pi \underbrace{t^3}_{y} \underbrace{t\sqrt{4+9t^2}\,dt}_{ds}$$

$$= 2\pi \int_0^1 t^4 \sqrt{4+9t^2}\,dt.$$

Ex. Revolve

$(x = t^2, \; y = t^3, \; 0 \le t \le 1)$ around the x-axis.



$$\int t^4 (4+9t^2)^{1/2}\, dt$$

$t = \frac{2}{3}\tan u$, $dt = \frac{2}{3}\sec^2 u\, du$

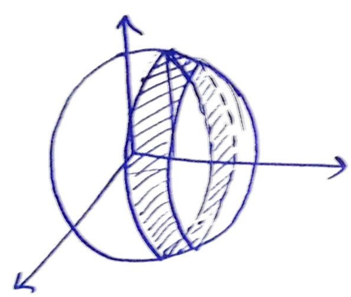
$\tan^2 u + 1 = \sec^2 u$

Putting all of this together gives us

$$\int t^4(4+9t^2)^{1/2}\, dt = \int \left(\frac{2}{3}\tan u\right)^4 \left(4+9\left(\frac{4}{9}\tan^2 u\right)\right)^{1/2} \left(\frac{2}{3}\sec^2 u\, du\right)$$

$$= \left(\frac{2}{3}\right)^5 \int \tan^4 u\, (2\sec u)\,(\sec^2 u\, du) \ldots$$

Ex.



For upper hemisphere,

$$y = \sqrt{1-x^2}$$

$$\int_{x=a}^{x=b} \pi y\, ds$$

$$= \int_{x=a}^{x=b} \pi\sqrt{1-x^2}\, \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \pi \int_{x=a}^{x=b} \frac{dx}{\cancel{\sqrt{1-x^2}}} = \pi(b-a).$$

Similarly, for the lower hemisphere,

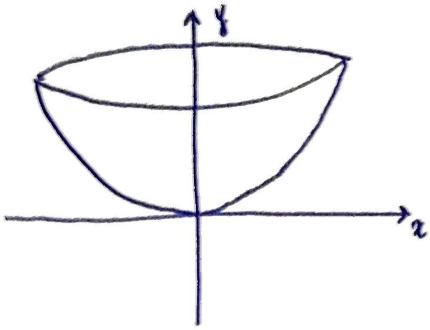
$y = -\sqrt{1-x^2}$, similarly $\pi(b-a)$.

Special cases: whole sphere: $a=-1$, $b=1$; 4π.

For half sphere, $a=0$ and $b=1$; surface area $= 2\pi$.

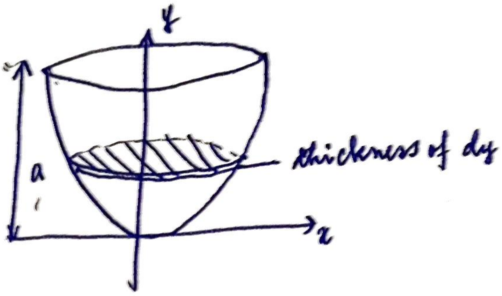
VOLUMES BY DISKS AND SHELLS

Examples A witch's cauldron



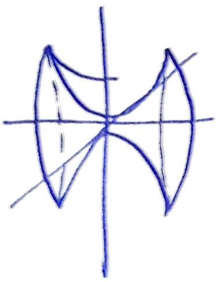
$y = x^2$ rotated along the y-axis

Method 1 Disks



The area of the surface of the disk in the figure is πx^2.

The disk has thickness dy and volume $dV = \pi x^2\, dy$.



The volume V of the cauldron is

$$V = \int_0^a \pi x^2\, dy \quad (\text{substitute } y = x^2)$$

$$= \int_0^a \pi y\, dy = \pi \left.\frac{y^2}{2}\right|_0^a = \frac{\pi a^2}{2}.$$

If a is 1 meter, then $V = \frac{\pi}{2} a^2$ gives

$$V = \frac{\pi}{2}\, m^3 = \frac{\pi}{2}(100\, cm)^3 = \frac{\pi}{2}\, 10^6\, cm^3 \approx 1600 \text{ litres} \quad (\text{a huge cauldron})$$

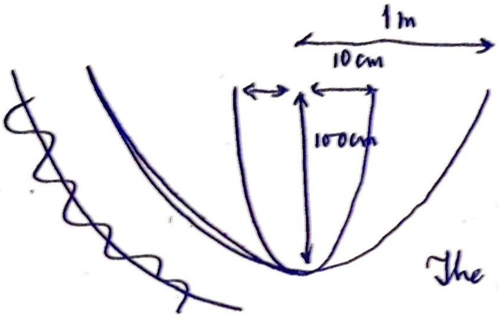
Warning about limits: If $a = 100\, cm$,

then $V = \frac{\pi}{2}(100)^2 = \frac{\pi}{2}\, 10^4\, cm^3 = \frac{\pi}{2}\, 10 \sim 16$ litres

But $100\, cm = 1$ meter. So, why is this answer different?

The resolution of the paradox is hiding in the equation $y = x^2$.

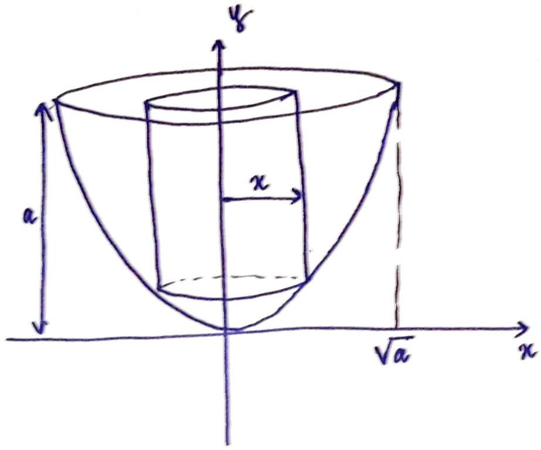
At the top, $100 = x^2 \Rightarrow x = 10$ cm. So the second cauldron looks like



By contrast, when $a = 1$ m, the top is 10 times wider: $1 = x^2$ or $x = 1$ m. Our equation, $y = x^2$, is not <u>scale-invariant</u>. The shape described depends upon the units used.

<u>Method 2 : Shells</u>

cylinder method



x : radius of the cylinder. Thickness of the cylinder $= dx$.

Height of the cylinder $= a - y = a - x^2$.

The thin shell / cylinder has height $a - x^2$, circumference $2\pi x$, and thickness dx.

$$dV = (a - x^2)(2\pi x)\,dx$$

$$V = \int_{x=0}^{x=\sqrt{a}} (a - x^2)(2\pi x)\,dx = 2\pi \int_0^{\sqrt{a}} (ax - x^3)\,dx$$

$$= 2\pi \left(a\frac{x^2}{2} - \frac{x^4}{4} \right)\Bigg|_0^{\sqrt{a}} = 2\pi \left(\frac{a^2}{2} - \frac{a^2}{4} \right) = 2\pi \frac{a^2}{4} = \frac{\pi a^2}{2}$$

(same as before)

Example. ~~Pipe flow~~ Boiling cauldron

~~Poiseuille was the first person to study fluid flow in pipes (arteries, capillaries).~~

Now, let's fill this cauldron with water, and light a fire under it to get the water to boil (at 100 °C). Let's say it's a cold day: the temperature of the air outside the cauldron is 0 °C. How much energy does it take to boil this water, i.e., to raise the water's temperature from 0 °C to 100 °C? Assume the temperature decreases linearly between the top and the bottom ($y=0$) of the cauldron: $T = (100 - 30y)\,°C$.

Use the method of disks, because the water's temperature is constant over each horizontal disk. The total heat required is

$$H = \int_0^1 T(\pi x^2)\,dy \quad \text{[units are degree . cubic meters]}$$

$$= \int_0^1 (100 - 30y)(\pi y)\,dy$$

$$= \pi \int_0^1 (100y - 30y^2)\,dy = \pi\left(50y^2 - 10y^3\right)\Big|_0^1 = 40\pi \text{ deg m}^3.$$