DOUBLE INTEGRATION

1. Find the $\underline{\text{mass}}$ of the region R bounded by

$y = x+1$; $y = x^2$; $x = 0$ and $x = 1$, if density $= \delta(x,y) = xy$

$\underline{\text{Soln}}$. Inner limits: y from x^2 to $x+1$.

Outer limits: x from 0 to 1.

$$M = \iint_R \delta(x,y)\, dA = \int_{x=0}^{x=1} \int_{y=x^2}^{y=x+1} xy\, dy\, dx$$

You know that

$$\boxed{\text{mass} = \underset{}{\overset{(\text{or area})}{\text{volume}}} \cdot \text{density}}$$

So if you integrate for volume (or area), while multiplying by density, that will give you mass.

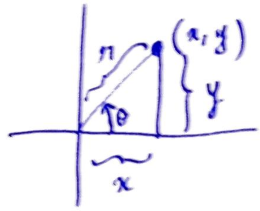
$$= \int_{x=0}^{x=1} \left(\frac{x^3}{2} + x^2 + \frac{x}{2} - \frac{x^5}{2} \right) dx = \frac{5}{8}.$$

POLAR COORDINATES AND JACOBIAN

1. Let $r = \sqrt{x^2+y^2}$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$. Directly calculate the Jacobian $\frac{\partial(r,\theta)}{\partial(x,y)} = \frac{1}{r}$.

$\underline{\text{Soln}}$. $$\boxed{\frac{\partial(r,\theta)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(r,\theta)} = 1}$$

$$\left[\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}\right.$$



$\cos\theta = \frac{x}{r}$ etc.

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{2x}{2\sqrt{x^2+y^2}} & \frac{2y}{2\sqrt{x^2+y^2}} \\ \frac{-y/x^2}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \end{vmatrix}$$

$$= \frac{x^2+y^2}{r^3} = \frac{1}{r}.$$

Again,

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \dfrac{\partial(r\cos\theta)}{\partial r} & \dfrac{\partial(r\sin\theta)}{\partial r} \\ \dfrac{\partial(r\cos\theta)}{\partial\theta} & \dfrac{\partial(r\sin\theta)}{\partial\theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & \sin\theta \\ (-\sin\theta)r & r\cos\theta \end{vmatrix}$$

$$= r(\cos^2\theta) + r\sin^2\theta = r.$$

2. For the change of variables $x = u$, $y = \sqrt{r^2 - u^2}$, write $dx\,dy$ in terms of u and r.

<u>Soln</u>. We know $dx\,dy = \left|\dfrac{\partial(x,y)}{\partial(u,r)}\right| du\,dr$

$$\frac{\partial(x,y)}{\partial(u,r)} = \begin{vmatrix} x_u & x_r \\ y_u & y_r \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ -\dfrac{u}{\sqrt{r^2-u^2}} & \dfrac{r}{\sqrt{r^2-u^2}} \end{vmatrix} = \frac{r}{\sqrt{r^2-u^2}}$$

Hence $dx\,dy = \dfrac{r}{\sqrt{r^2-u^2}}\,du\,dr$.