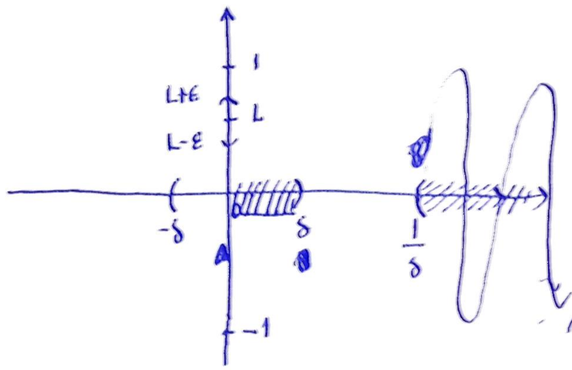
Showing that $\lim_{x \to 0} \sin \frac{1}{x}$ does not exist.



Let's say $\lim_{x \to 0} \sin \frac{1}{x}$ exists and equals $L \in [0,1]$. For ~~it~~ to exist, given any $\varepsilon > 0$, we must be able to show existence of $\delta > 0$ such that ~~x ∈~~

$$x \in (-\delta, \delta) \Rightarrow \sin \frac{1}{x} \in (L-\varepsilon, L+\varepsilon).$$

$$(-\delta, \delta) \longrightarrow \left(-\infty, -\frac{1}{\delta}\right) \cup \left(\frac{1}{\delta}, \infty\right) \longrightarrow [0,1].$$

$$x \xmapsto{\varphi} \frac{1}{x} \xmapsto{\psi} \sin \frac{1}{x}$$

So, $-\delta < x < \delta$

$\Rightarrow -\delta < x \leq 0 \Rightarrow \frac{1}{x} < -\frac{1}{\delta}$

$0 \leq x < \delta \Rightarrow \frac{1}{x} > \frac{1}{\delta}$

So, $\varphi(x) = \frac{1}{x}$ sends segment $(-\delta, 0]$ to say $(-\infty, -\frac{1}{\delta})$ and $[0, \delta)$ to $(\frac{1}{\delta}, \infty)$.

~~$x \in (-\delta, 0]$~~ $\sin \frac{1}{x}$ for $x \in (-\delta, 0]$ amounts to $\sin y$ for $y \in (-\infty, \frac{1}{\delta})$

and $\sin \frac{1}{x}$ for $x \in [0, \delta)$ amounts to $\sin y$ for $y \in (\frac{1}{\delta}, \infty)$.

But sine function is periodic and in $(-\infty, \frac{1}{\delta}) \cup (\frac{1}{\delta}, \infty)$, that has infinite length, takes every value in $[0,1]$, and does not lie within $(L-\varepsilon, L+\varepsilon)$. So, $\lim_{x \to 0} \sin \frac{1}{x}$ does not exist.

(This holds for any $\delta > 0$)