Functions of several variables

Recall: For a fn of one variable we can plot its graph and the derivative is the slope of the tangent line to the graph.

Plotting graphs of 2 variables: examples $z = -y$, $z = 1 - x^2 - y^2$; using slices by the coordinate planes.

<u>Contour plot</u>: level curves $f(x,y) = c$, amounts to slicing the graph by horizontal planes $z = c$.

$z = x^2 + y^2$, $z = y^2 - x^2$.

Contour plot gives some qualitative info about how f varies when we change x, y.

Partial Derivatives

$$f_x = \frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x} \quad ; \text{ same for } f_y$$

<u>Geometric interpretation</u>: f_x, f_y are slopes of tangent lines of vertical slices of the graph of f (fixing $y = y_0$; fixing $x = x_0$)

How to compute: treat x as variable, y as constant.

<u>Example</u>: $f(x,y) = x^3 y + y^2$ then $f_x = 3x^2 y$, $f_y = x^3 + 2y$.

~~PS~~ Linear approximation

Interpretation of f_x, f_y as slopes of slices of the graph by planes parallel to xz and yz planes.

Linear approximation formula: $\Delta f \approx f_x \Delta x + f_y \Delta y$.

Justification. f_x and f_y give slopes of two lines tangent to the graph:

$y = y_0$, $z = z_0 + f_x(x_0, y_0)(x - x_0)$ and $x = x_0$, $z = z_0 + f_y(x_0, y_0)(y - y_0)$.

We can use this to get the equation of the tangent plane to the graph:

$$z = z_0 + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Approximation formula = the graph is close to its tangent plane.

Min / max problems

At a local max or min, $f_x = 0$ and $f_y = 0$ (since (x_0, y_0) is a local max or min of the slice). Because 2 lines determine tangent plane, this is enough to ensure that tangent plane is horizontal (approximation formula: $\Delta f \simeq 0$ or rather $|\Delta f| \ll |\Delta x|, |\Delta y|$).

Def of critical point: (x_0, y_0) where $f_x = 0$ and $f_y = 0$.

A critical point may be a local min, local max or saddle.

Example: $f(x,y) = x^2 - 2xy + 3y^2 + 2x - 2y$

Critical point: $f_x = 2x - 2y + 2 = 0$, $f_y = -2x + 6y - 2 = 0$ gives $(x_0, y_0) = (-1, 0)$.

(only one critical point)

Observe $f = (x-y)^2 + 2y^2 + 2x - 2y = (x - y + 1)^2 + 2y^2 - 1 \geq -1$, so min.

Least squares. experimental data (x_i, y_i) $(i = 1, \dots, n)$ want to find a best-fit line $y = ax + b$.

(The unknowns here are a, b and not x, y).

Deviations: $y_i - (ax_i + b)$, want to minimize the total square deviation

$$D = \sum_i (y_i - (ax_i + b))^2$$

* $\frac{\partial D}{\partial a} = 0$ and $\frac{\partial D}{\partial b} = 0$ leads to a 2×2 linear system for a and b

$$\left(\sum x_i^2\right) a + \left(\sum x_i\right) b = \sum x_i y_i$$

$$\left(\sum x_i\right) a + nb = \sum y_i$$

Least-squares setup also works in other cases, e.g., exponential laws $y = ce^{ax}$ (taking logarithms: $\ln y = \ln c + ax$, so setting $b = \ln c$ we reduce to linear case), or quadratic laws $y = ax^2 + bx + c$ (minimizing total sq. deviation leads to a 3×3 linear system for a, b, c).

Moore's law.

Second derivative test

Let (x_0, y_0) be a critical point of $f(x,y)$ and A, B and C be ~~[illegible]~~ computed as

$$A = (f_{xx})\Big|_{x=x_0}, \quad B = (f_{xy})\Big|_{\substack{x=x_0\\ y=y_0}}, \quad C = (f_{yy})\Big|_{y=y_0}.$$

$$= (f_{yx})\Big|_{x=x_0,\, y=y_0}$$

this equality has some intricacies about it; the discussion in theory class.

Then,

$AC - B^2 > 0$, $A > 0$ or $C > 0 \Rightarrow (x_0, y_0)$ is a minimum point.

$AC - B^2 > 0$, $A < 0$ or $C < 0 \Rightarrow (x_0, y_0)$ is a maximum point.

$AC - B^2 < 0 \Rightarrow (x_0, y_0)$ is a saddle point.

If $AC - B^2 = 0$, the test fails.