Solutions to Problem Set-1

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1. Prove that $\sqrt{5}$ is an irrational number.

Soln. Suppose, on the contrary, that $\sqrt{5}$ is a rational number, and can be written as $\sqrt{5} = \dfrac{p}{q}$ such that $p, q \in \mathbb{Z}$; $\gcd(p, q) = 1$. Then $p = q\sqrt{5}$, i.e.,

$$p^2 = 5q^2. \tag{1}$$

Right-hand side is divisible by 5, so 5 divides p^2, i.e., 5 divides p. Hence, 25 divides p^2, the left-hand side of (1). So 25 must divide $5q^2 \Rightarrow 5$ divides q^2, which means 5 divides q. Then $\gcd(p, q) \geq 5$, a contradiction.

So our initial assumption was wrong and $\sqrt{5}$ cannot be expressed as a rational number.

2. Prove that $\mathbb{N}$ is not bounded above.

Soln. Suppose there exists $x \in \mathbb{R}$ such that $x > n\ \forall\, n \in \mathbb{N}$, i.e., x is an upper bound of $\mathbb{N}$. By Least Upper Bound Property, there exists some $M \in \mathbb{R}$ such that $M = \sup \mathbb{N}$. Then,

$$\begin{aligned} & n \leq M\ \forall\, n \in \mathbb{N} \\ \Rightarrow\ & n - 1 \leq M - 1\ \forall\, n \in \mathbb{N} \\ \Rightarrow\ & (n+1) - 1 \leq M - 1\ \forall\, n \in \mathbb{N} \text{ (recall Peano's axioms)} \\ \Rightarrow\ & n \leq M - 1\ \forall\, n \in \mathbb{N}. \end{aligned}$$

Therefore $M - 1$ is also an upper bound of $\mathbb{N}$ but M is supposed to be the least upper bound, a contradiction.

3. Let $A \subseteq \mathbb{R}$ be bounded. Prove that $\sup(-A) = -\inf A$ and $\inf(-A) = -\sup A$.

Soln. The point of this exercise is for you to understand why it is not required to include the existence of infimum in the axiom of completeness.

I shall show you how $-\sup(-A) = \inf A$. A is bounded, which means that there exists $x \in \mathbb{R} \ni x \leq a \Rightarrow -x \geq -a \Rightarrow x$ is an upper bound for $-A = \{-a \mid a \in A\}$. Here, x has been an arbitrary lower bound for A. By the axiom of completeness as we know it, $\exists\, y \in \mathbb{R} \ni y = \sup(-A)$, i.e., $-a \leq y \leq -x\ \forall\, a \in A \Rightarrow x \leq -y \leq a$. Here, $-y$ is an lower bound for A, and for any arbitrary $x \leq a$, greater than x, i.e., the greater lower bound or infimum of A. Therefore, $-\sup(-A) = \inf A$.

You should be able to show the other part by replacing A by $-A$.

4. Find the superma and infima of the following sets:

(a) $\left\{x \in \mathbb{R} \mid 3x^2 + 8x - 3 < 0\right\}$.

Soln. $3x^2+8x-3<0 \Rightarrow 3x^2+9x-x-3<0 \Rightarrow (3x-1)(x+3)<0$. Either $3x-1>0, x+3<0 \Rightarrow x>\frac{1}{3}, x<3$ which is not possible together, or $3x-1<0, x+3>0 \Rightarrow x<\frac{1}{3}, x>-3$. So $x \in \left(-3, \frac{1}{3}\right)$. Thus, $\sup x = \frac{1}{3}, \inf x = -3$.

(b) $\left\{\frac{1}{n}+\frac{1}{m} \,\middle|\, n, m \in \mathbb{N}\right\}$.

Soln. Check with $n=1, m\in\mathbb{N}$: $\frac{1}{1}+\frac{1}{1}=2, \frac{1}{1}+\frac{1}{2}=1\frac{1}{2}, \frac{1}{1}+\frac{1}{3}=1\frac{1}{3}$... (decreasing)

Check with $n=2, m\in\mathbb{N}$: $\frac{1}{2}+\frac{1}{1}=1\frac{1}{2}, \frac{1}{2}+\frac{1}{2}=1, \frac{1}{2}+\frac{1}{3}=\frac{5}{6}$... (decreasing)

and so on.

You can see that the supremum has been attained at $n=1, m=1$. But what about the infimum? Take any positive real number <1, say ε. By Archimedean property, we can find $n\in\mathbb{N}$ such that $n>\frac{2}{\varepsilon}\in\mathbb{R}$ and take $m=n$. We see that $\frac{1}{n}+\frac{1}{m}<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$. So there exist natural numbers n, m for which $\frac{1}{n}+\frac{1}{m}$ is less than any arbitrary positive real number, but is never negative. What is the infimum of positive real numbers?

(c) $\left\{n\in\mathbb{N} \,\middle|\, \frac{n-1}{n+1}\cos\frac{2n\pi}{3}\right\}$.

Soln.

$$\begin{aligned}
\cos\frac{2.1\pi}{3} &= 2\cos^2\frac{\pi}{3}-1 = 2.\frac{1}{4}-1=-\frac{1}{2},\\
\cos\frac{2.2\pi}{3} &= 2\cos^2\frac{2\pi}{3}-1 = 2.\frac{1}{4}-1=-\frac{1}{2},\\
\cos\frac{2.3\pi}{3} &= \cos(2\pi)=\cos 0 = 1,\\
\cos\frac{2.4\pi}{3} &= \cos\left(2\pi+\frac{2.1\pi}{3}\right) \text{ etc.}
\end{aligned}$$

Since $\cos\frac{2n\pi}{3}$ is a negative constant, the infimum and supremum of $\left\{n\in\mathbb{N} \,\middle|\, \frac{n-1}{n+1}\cos\frac{2n\pi}{3}\right\}$ are attained respectively at the supremum and infimum of $\left\{n\in\mathbb{N} \,\middle|\, \frac{n-1}{n+1}\right\}$.

For $n=1$, $\frac{n-1}{n+1}=0$.

For $n=2$, $\frac{n-1}{n+1}=\frac{1}{3}$.

For $n=3$, $\frac{n-1}{n+1}=\frac{2}{4}$

For $n=4$, $\frac{n-1}{n+1}=\frac{3}{5}$

For $n=5$, $\frac{n-1}{n+1}=\frac{4}{6}$

For $n=6$, $\frac{n-1}{n+1}=\frac{5}{7}$ etc.

Correspondingly,

for $n=1$, $\frac{n-1}{n+1}\cos\frac{2n\pi}{3}=0.\left(-\frac{1}{2}\right)=0$;

for $n=2$, $\frac{n-1}{n+1}\cos\frac{2n\pi}{3}=\frac{1}{3}\left(-\frac{1}{2}\right)-=-\frac{1}{6}$;

for $n = 3$, $\frac{n-1}{n+1}\cos\frac{2n\pi}{3} = \frac{2}{4}.1 = \frac{1}{2}$;
for $n = 4$, $\frac{n-1}{n+1}\cos\frac{2n\pi}{3} = \frac{3}{5}\left(-\frac{1}{2}\right) = -\frac{3}{10}$;
for $n = 5$, $\frac{n-1}{n+1}\cos\frac{2n\pi}{3} = \frac{4}{6}\left(-\frac{1}{2}\right) = -\frac{1}{3}$;
for $n = 6$, $\frac{n-1}{n+1}\cos\frac{2n\pi}{3} = \frac{5}{7}.1 = \frac{5}{7}$ etc.

[I am using the terminology of sequences here. Strictly speaking, this example can be studied without knowing about sequences; yet since you have studies about sequences in class, it won't harm to use one or two words that ease the arguments considerably.]

Let $a_n = \frac{n-1}{n+1}\cos\frac{2n\pi}{3}$. Then absolute values of the entries in each of the 3 sequences $\left\{a_{3k}\right\}_{k\in\mathbb{N}}$, $\left\{a_{3k+1}\right\}_{k\in\mathbb{N}}$ and $\left\{a_{3k+2}\right\}_{k\in\mathbb{N}}$ increase with increasing k. This leaves us with $a_2 = -\frac{1}{6}$ as the least value in a discrete setting, giving the infimum. All these a_n's are less than 1. Consider the subsequence $\left\{a_{3k+2}\right\}_{k\in\mathbb{N}}$ that you can actually see to tend to 1 as $k \to \infty$. So, we can be sure that 1 is the supremum.

(d) $\left\{ n \in \mathbb{N} \;\middle|\; \frac{(n+1)^2}{2^n} \right\}$.

Soln. Let $a_n = \frac{(n+1)^2}{2^n}$. Then $\frac{a_n}{a_{n+1}} = \frac{2\left(n^2+2n+1\right)}{n^2+4n+4} = \frac{2\left(n^2+4n+4\right)}{n^2+4n+4} - \frac{2(2n+3)}{n^2+4n+4} = 2 - \frac{4n+6}{n^2+4n+4} > 1$. So we have $a_{n+1} < a_n$ for all $n \geq 3$. $a_1 = 2, a_2 = \frac{9}{4}, a_3 = 2, a_4 = \frac{25}{16}$ etc. The highest is attained at $n = 2$. The infimum is 0 since 2^n values increase with a much higher rate with increasing n, than $(n+1)^2$. We have seen that $a_{n+1} < \frac{2}{3}a_n$, so for any given $\varepsilon > 0$ we can find $n \in \mathbb{N}$ such that $a_n < \varepsilon$.

5. Prove that $[0, 1]$ is uncountable.

Soln. Consider numbers written like this $\overline{0.x_1x_2x_3\ldots}$ where the entries after the decimal point are either 0 or 1. It is clear that their collection is a proper subset of $[0, 1]$. Check out Rudin's book, Theorem 2.14 (I have explained this in class), which shows that the collection of such numbers is uncountable. $[0, 1]$, being a superset, is certainly so.

6. Find the sets of all interior points for the following sets: $\mathbb{Q}$, $(0, 2]$.

Soln. For $\mathbb{Q}$, Let's say $q \in \mathbb{Q}$. Consider any $\varepsilon > 0$ and the open interval $(q-\varepsilon, q+\varepsilon)$. Since irrational numbers are dense in $\mathbb{R}$, there exists some $r \notin \mathbb{Q}$ such that $q - \varepsilon < r < q$, i.e., $r \in (q-\varepsilon, q+\varepsilon)$. So q has no interior point.

For $(0, 2]$, If $2 \neq x \in (0, 2]$, then there exists some $\varepsilon_x > 0$ such that $\left(x - \varepsilon_x, x + \varepsilon_x\right) \subset (0, 2]$. So, $(0, 2)$ is the interior.

7. Let S be a non-empty bounded set. Consider the set $T = \{|x-y| : x, y \in S\}$. Prove that T is bounded above. Find the supremum of T.

Soln. S is a bounded set. So there exists $M \in \mathbb{R}_{\geq 0}$ such that for any $x \in S$, $|x| < M$. By triangle inequality, $|x-y| \leq |x| + |y| < 2M$. So, T is bounded.

$\sup T = |\sup S - \inf S|$.