Problem set 2

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Note 1 *Writing solutions like this in the exams might not fetch you full marks. Please clearly mention the name of any theorem or result you use; and also try to be very specific with ε-δ rigourously in each problem. Don't write one step without proper literal justification from the definition, unless prompted otherwise.*

1. Using definition, prove convergence and find the limits.

1(a) $\left\{\log\left(1+\frac{1}{n^2}\right)\right\}_{n\in\mathbb{N}}$.

Soln. In math class, always assume base e unless mentioned otherwise. How to do it: you know if it's convergent, the limit has to be $\log 1 = 0$. Given $\varepsilon > 0$, $\left|\log\left(1+\frac{1}{n}\right) - 0\right| < \varepsilon \Leftrightarrow 1 + \frac{1}{n} < e^{\varepsilon} \Leftrightarrow \frac{1}{n} < e^{\varepsilon} - 1 \Leftrightarrow n > \frac{1}{e^{\varepsilon}-1}$. Thus, 0 is the limit.

1(b) $\left\{(-1)^n \frac{1}{n}\right\}_{n\in\mathbb{N}}$.

Soln. $n > \varepsilon \Leftrightarrow \left|(-1)^n \frac{1}{n} - 0\right| < \varepsilon$.

1(c) $\left\{\frac{5n-1}{2n+1}\right\}_{n\in\mathbb{N}}$.

Soln. $n > \frac{1}{4}\left(\frac{7}{\varepsilon} - 2\right) \Leftrightarrow \left|\frac{5n-1}{2n+1} - \frac{5}{2}\right| < \varepsilon$.

1(d) $\left\{a^{\frac{1}{n}}\right\}_{n\in\mathbb{N}}$, $a > 0$.

Soln. *Case 1*: $a \geq 1$. Then $\left|a^{\frac{1}{n}} - 1\right| < \varepsilon \Leftrightarrow n > \log_a(\varepsilon + 1)$. The complementary case follows similarly.

2. Let $\{a_n\}_{n\in\mathbb{N}}$ be a convergent sequence of real numbers such that there exists $k \in \mathbb{R}$ such that $a_n > 0$ for all $n \geq k$. Prove that $\lim a_n \geq 0$.

Soln. Let $\lim a_n = -m < 0$. Then $\exists\ N \in \mathbb{N}$ such that $|a_n - (-m)| < m$ for all $n \geq N$, i.e., $|a_n + m| < m$, i.e., $a_n < 0$, a contradiction.

3. Suppose $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ be two sequences such that $\{a_n + b_n\}_{n\in\mathbb{N}}$ is convergent. Does that mean they are convergent individually?

Soln. No! Let $a_n = n \; \forall \; n \in \mathbb{N}$, $b_n = -n \; \forall \; n \in \mathbb{N}$. Then $\left\{a_n + b_n\right\}_{n\in\mathbb{N}} = \{0\}_{n\in\mathbb{N}}$ which is convergent.

4. Deduce whether the following sequences are convergent. Find the limit if convergent.

4(a) $\sqrt[n]{\sum_{i=1}^{k} a_i^n}$ where $0 \le a_1 \le a_2 \le \cdots \le a_k$.

Soln. $0 \le \sqrt[n]{\sum_{i=1}^{k} a_i^n} \le \sqrt[n]{n a_k^n} = \sqrt[n]{n} a_k$, so depends on a_k.

4(b) $\left\{\frac{1}{n^2} \sin \frac{1}{n}\right\}_{n\in\mathbb{N}}$

Soln. $0 \le \frac{1}{n^2} \sin \frac{1}{n} \le \frac{1}{n^2}.1 \to 0$ as $n \to \infty$.

4(c) $\left\{\sqrt{n+1} - \sqrt{n}\right\}_{n\in\mathbb{N}}$.

Soln. $\sqrt{n+1} - \sqrt{n} = \dfrac{\left(\sqrt{n+1} - \sqrt{n}\right)\left(\sqrt{n+1} + \sqrt{n}\right)}{\sqrt{n+1} + \sqrt{n}} = \dfrac{1}{\sqrt{n+1} + \sqrt{n}} < \dfrac{1}{\sqrt{n} + \sqrt{n}} = \dfrac{1}{2\sqrt{n}} \xrightarrow[n\to\infty]{} 0$.

4(d) $\{\sin(n!\alpha\pi)\}_{n\in\mathbb{N}}$ where $\alpha \in \mathbb{Q}$.

Soln. Let $\alpha = \frac{p}{q}$, $q \in \mathbb{N}$. Then, for $n = q$ onwards, we have that $n!\alpha\pi$ is a multiple of 2π because 2 divides $(q-1)!$. Thus, after that the sequence is identically 0 which converges to 0.

5(a) Prove convergence and find the limit: $\left\{\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \dots\right\}$.

Soln. Let the limit be K. Then, at K the sequence can be visualized to have sort of "stabilized", i.e., $K = \sqrt{2K} \Rightarrow K^2 - 2K = 0 \Rightarrow K(K-2) = 0$. Clearly, $K > \sqrt{2}$, so $K \neq 0$; hence $K = 2$.

5(b) Prove convergence and find the limit: $\left\{\sqrt{7}, \sqrt{7+\sqrt{7}}, \sqrt{7+\sqrt{7+\sqrt{7}}}, \dots\right\}$.

Soln. Similar. Limit L should be such that $L = \sqrt{7+L} \Rightarrow L^2 = 7 + L \Rightarrow L^2 - L - 7 = 0 \Rightarrow L = \dfrac{1 \pm \sqrt{29}}{2}$. But $L > 7 \Rightarrow L = \dfrac{1+\sqrt{29}}{2}$.

6(a) Cauchy or not? $\left\{\frac{n+1}{n-2}\right\}_{n\in\mathbb{N}}$.

Soln. Back-calculation: $\left|\frac{n+1}{n-2} - \frac{m+1}{m-2}\right| < \varepsilon \Leftrightarrow \left|\frac{1}{n-2} - \frac{1}{m-2}\right| < \frac{\varepsilon}{3}$. Now, $\left|\frac{1}{n-2} - \frac{1}{m-2}\right| \le \left|\frac{1}{n-2}\right| + \left|\frac{1}{m-2}\right| \le \left|\frac{2}{n-2}\right|$ if $m \ge n$. Thus, for $m, n \ge 2$, $n, m > \frac{6}{\varepsilon} - 2 \Rightarrow \left|\frac{1}{n-2} - \frac{1}{m-2}\right| < \frac{\varepsilon}{3}$. Cauchy.

6(b) Cauchy or not? $\left\{ \sum_{k=1}^{n} \frac{\sin k!}{k(k+1)} \right\}_{n\in\mathbb{N}}$.

Soln.

$$\begin{aligned} \left| \sum_{k=1}^{n} \frac{\sin k!}{k(k+1)} \right| &\le \sum_{k=1}^{n} \left| \frac{1}{k(k+1)} \right| \\ &\le \sum_{k=1}^{n} \left| \frac{1}{k^2} \right| \ (n > m) \end{aligned}$$

which is convergent. So, the given sequence is convergent, hence Cauchy.

6(c) Cauchy or not? $\left\{ 1 + \frac{1}{2} + \cdots + \frac{1}{n} \right\}_{n\in\mathbb{N}}$.

Soln. Not convergent, so not Cauchy.

7. Identify which of these are subsequences of $\left\{ \frac{1}{n} \right\}_{n\in\mathbb{N}}$.

First let us recall what it means to be a subsequence of $\left\{ \frac{1}{n} \right\}_{n\in\mathbb{N}}$. That the sequence has to be of the form $\left\{ \frac{1}{f(n)} \right\}_{n\in\mathbb{N}}$ where $f : \mathbb{N} \to \mathbb{N}$ should be an increasing function.

7(a) $\left\{ \frac{1}{2n-1} \right\}_{n\in\mathbb{N}}$

Soln. $f(n) = 2n - 1$.

7(b) $\left\{ \frac{1}{2^n} \right\}_{n\in\mathbb{N}}$

Soln. $f(n) = 2^n$.

7(c) $\left\{ \frac{1}{\log(n+1)} \right\}_{n\in\mathbb{N}}$

Soln. $\log(n+1)$ doesn't even need to be an integer.

7(d) $\left\{ \frac{1}{3 + \sin\frac{n\pi}{2}} \right\}_{n\in\mathbb{N}}$

Soln. $f' : \mathbb{N} \to \mathbb{N}; n \mapsto 3 + \sin\frac{n\pi}{2}$ is, however, not an increasing function. So, no.

8. If the subsequences $\{u_{2n}\}_{n\in\mathbb{N}}$ and $\{u_{2n-1}\}_{n\in\mathbb{N}}$ of $\{u_n\}_{n\in\mathbb{N}}$ are convergent, then prove that $\{u_n\}_{n\in\mathbb{N}}$ is convergent.

Soln. $\left\{u_{2n}\right\}_{n\in\mathbb{N}}$ is convergent means that for all $\varepsilon > 0$ there exists $N_1 \in \mathbb{N}$ such that $u_{2n} < \varepsilon \ \forall\, n > N$.

$\left\{u_{2n-1}\right\}_{n\in\mathbb{N}}$ is convergent means that for all $\varepsilon > 0$ there exists $N_2 \in \mathbb{N}$ such that $u_{2n} < \varepsilon \ \forall\, n > N$.

Take $N = \max\left\{N_1, N_2\right\}$ and combine the statements for u_n.

8. Give an example of a sequence which has two convergent subsequences but is itself not convergent.

Soln. *Because the limits are different*: One such example might be $\left\{(-1)^n\right\}_{n\in\mathbb{N}}$. Subsequences $\left\{(-1)^{2n-1} = -1\right\}_{n\in\mathbb{N}}$ and $\left\{(-1)^2 n = 1\right\}_{n\in\mathbb{N}}$ converge respectively to -1 and 1.

Because there exists a third divergent subsequence: Let's consider $\left\{n \sin \frac{n\pi}{2}\right\}_{n\in\mathbb{N}}$. We get two convergent subsequences for $4\mathbb{N}$ $\left[\{4n \sin 2n\pi\}_{n\in\mathbb{N}}\right]$ and $4\mathbb{N} + 2$ $\left[\{(4n+2)\sin(2n+1)\pi\}_{n\in\mathbb{N}}\right]$ but get a divergent subsequence for $2\mathbb{N} - 1$ $\left[\left\{(2n-1)\sin\frac{(2n-1)\pi}{2}\right\}_{n\in\mathbb{N}}\right]$.