3. (i) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^x}$

$$a_n = \frac{1}{n^x} \Rightarrow 2^n a_{2^n} = 2^n \frac{1}{(2^n)^x} = \frac{1}{(2^n)^{x-1}}$$

$\cancel{\frac{1-n}{1+2n}}$

If $x > 1$, the series converges.

If $x = 1$, the series diverges.

If $x < 1$, the series obviously diverges.

(ii) $$\sum_{n=2}^{\infty} \frac{1}{n \log n^x} = \sum_{n=2}^{\infty} \frac{1}{nx \log n}.$$

Since $x \in \mathbb{R}$ is a constant, the series always diverges. See solution of 8 (iii) and try to show yourself.

(iii) $$\sum_{n=3}^{\infty} \frac{1}{\log n^x} = \sum_{n=3}^{\infty} \frac{1}{x \log n} = \frac{1}{x} \sum_{n=3}^{\infty} \frac{1}{\log n}, \text{ diverges,}$$

since $x \in \mathbb{R}$ is a constant.

4. Convergent or divergent:

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{n^4}{n!}$

Soln. $$a_n = \frac{n^4}{n!} \Rightarrow \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)^4}{n^4} \cdot \frac{n!}{(n+1)!}\right| = \left|\left(1+\frac{1}{n}\right)^4 \frac{1}{(n+1)}\right|$$

$\to 0$

4. (ii) $\displaystyle\sum_{n=1}^{\infty} \frac{3}{4+2^n}$

Soln. $\displaystyle\sum_{n=1}^{\infty} \frac{3}{4+2^n} = 3\sum_{n=1}^{\infty} \frac{1}{4+2^n} \leq 3\sum_{n=1}^{\infty} \frac{1}{2^n} = 3\,\frac{1}{1-\frac{1}{2}}\cdot\frac{1}{2} = 3.$

4. (iii) $\displaystyle\sum_{n=1}^{\infty} \frac{1+n}{1+n^2}$

~~Soln.~~ $\cancel{\displaystyle\sum_{n=1}^{\infty} \frac{1+n}{1+n^2} \leq \sum_{n=1}^{\infty} \frac{1+n}{n^2} = \sum_{n=}}$

Soln. $\displaystyle 0 \leq \sum_{n=1}^{\infty} \frac{1+n}{1+n^2} \geq \sum_{n=1}^{\infty} \frac{n}{1+n^2} \geq \sum_{n=1}^{\infty} \frac{n}{n+n^2} = \sum_{n=1}^{\infty} \frac{1}{1+n}$,

which diverges

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n}$

When $x \geqslant 1$, this series diverges. For $x < 1$,

$$\sum_{n=1}^{\infty} \frac{x^n}{n} \leqslant \sum_{n=1}^{\infty} x^n = \frac{1}{1-x} \text{ (geometric series sum).}$$

(ii) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(\log n)^n}$.

For $n \geqslant \lfloor e^2 \rfloor$, $\log n \geqslant 2$. Then $(\log n)^n \geqslant 2^n$, i.e., $\dfrac{1}{(\log n)^n} \leqslant \dfrac{1}{2^n}$

$\therefore \displaystyle\sum_{n=\lfloor e^2 \rfloor}^{\infty} \frac{1}{(\log n)^n} \leqslant \sum_{n=\lfloor e^2 \rfloor}^{\infty} \frac{1}{2^{\infty}}$ which is convergent, and the given series is just this series plus a finite no. of terms.

(iii) $\displaystyle\sum_{n=1}^{\infty} \frac{\left(1+\frac{1}{n}\right)^{2n}}{e^n}$

$$a_n = \frac{\left(1+\frac{1}{n}\right)^{2n}}{e^n} \rightarrow \frac{e^{2}}{e^n} = e^{2-n}$$

$$\frac{a_{n+1}}{a_n} = \frac{\left(1+\frac{1}{n+1}\right)^{2n+2}}{e^{n+1}} \cdot \frac{e^n}{\left(1+\frac{1}{n}\right)^{2n}} = \frac{\left(\frac{n+2}{n+1}\right)^{2n+2}}{\left(\frac{n+1}{n}\right)^{2n}} \frac{1}{e}$$

$$= \left(\frac{(n+2)^{2n+2}}{(n+1)^{4n+2}} n^{2n}\right) \cdot \frac{1}{e} = \left(\frac{n+2}{n+1}\right)^{2n+2} \left(\frac{n}{n+1}\right)^{2n} \frac{1}{e} \rightarrow \frac{1}{e}.$$

$$\rightarrow \frac{e^2}{e^{n+1}} \cdot \frac{e^n}{e^2} = \frac{1}{e} < 1.$$

So by ratio test the series converges.

5. (iii) $\sum_{n=1}^{\infty} \frac{e^{-n}}{n^2} \leq \sum_{n=1}^{\infty} e^{-n}$ converges.

6. (i) $\sum_{n=1}^{\infty} \frac{n^n}{n!}$

~~Stirling's approximation~~

$$a_n = \frac{n^n}{n!}$$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1}}{(n+1)!} \frac{n!}{n^n} = \left(\frac{n+1}{n}\right)^n \frac{n+1}{n+1} = \left(1+\frac{1}{n}\right)^n \longrightarrow e > 1$$

diverges

$\cancel{(iii) \sum_{n=\alpha}^{\infty} \frac{x^n}{n!} < \sum_{n=\alpha}^{\infty} \frac{n^n}{n!} \quad x \geq 1.}$

(a) $\cancel{b}$ $a_n = \frac{x^n}{n!}$

$$\frac{a_{n+1}}{a_n} = \frac{x^{n+1}}{x^n} \cdot \frac{n!}{(n+1)!} = \frac{x}{n+1} \to 0 \quad \text{as } n \to \infty.$$

converges.

8. Verify whether the following series are convergent:

(i) $\sum_{n=0}^{\infty} \sin \frac{\pi}{2^n}$

<u>Soln.</u> $\sin x < x \quad \forall\, x \in \mathbb{R}_+$

$$\Rightarrow 0 < \sin \frac{\pi}{2^n} < \frac{\pi}{2^n} \quad \forall\, n \in \mathbb{N} \cup \{0\}$$

$$\Rightarrow \underbrace{0 < \sum_{n=0}^{\infty} \sin \frac{\pi}{2^n}}_{\text{convergent}} < \sum_{n=0}^{\infty} \frac{\pi}{2^n} = \pi \sum_{n=0}^{\infty} \frac{1}{2^n} = \pi \frac{1}{1-\frac{1}{2}} \quad \text{(geometric series)}$$

$$= 2\pi$$

(ii) $\sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{1}{\sqrt{n}}$

<u>Soln.</u> $\sin \frac{1}{\sqrt{n}} < \frac{1}{\sqrt{n}} \quad \forall\, n \in \mathbb{N}$.

Since all the terms of the given series are positive,

$$\underbrace{0 < \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{1}{\sqrt{n}}}_{\text{convergent}} \leq \sum_{n=1}^{\infty} \frac{1}{n} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}, \quad \text{convergent}$$

(iii) $\sum_{n=2}^{\infty} \frac{1}{n(\log(n^3))}$

Soln. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\log(n^3))} = \sum_{n=2}^{\infty} \frac{1}{n.3.\log n} = \frac{1}{3}\sum_{n=2}^{\infty} \frac{1}{n\log n}$

Condensation test : $a_n = \dfrac{1}{n\log n} \Rightarrow 2^n a_{2^n} = \dfrac{2^n}{2^n \log 2^n} = \dfrac{1}{n\log 2}$

$\therefore \displaystyle\frac{1}{3}\sum_{n=2}^{\infty} \frac{1}{n\log n}$ diverges.

(iv) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin\frac{n\pi}{2}}{n}$

Soln. $n = 4k \quad (k \in \mathbb{N}) \qquad \sin\frac{n\pi}{2} = \sin(2k\pi) = 0$

$n = 4k+1 \quad (k \in \mathbb{N}\cup\{0\}) \qquad \sin\frac{n\pi}{2} = \sin\frac{\pi}{2} = 1$

$n = 4k+2 \quad (k \in \mathbb{N}\cup\{0\}) \qquad \sin\frac{n\pi}{2} = \sin\pi = 0$

$n = 4k+3 \quad (k \in \mathbb{N}\cup\{0\}) \qquad \sin\frac{n\pi}{2} = \sin\frac{3\pi}{2} = -1.$

$$\therefore \sum_{n=1}^{\infty} \frac{\sin\frac{n\pi}{2}}{n} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1}$$

$\displaystyle\lim_{n\to\infty} \frac{1}{2n+1} = 0$ and $\left\{\frac{1}{2n+1}\right\}_{n\in\mathbb{N}} \downarrow$

$\therefore$ By alternating series test, $\displaystyle\sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi}{2}\right)}{n}$ converges.