4b

Let $\varepsilon > 0$; $x, y \in \mathbb{R}_+$.

We want $\big|\,|\sin x| - |\sin y|\,\big| < \varepsilon$ for $|x-y| < \delta$

Case 1, $\sin x > 0$, $\sin y > 0$.

$$\left|\sin x - \sin y\right| = \left|2\cos\frac{x+y}{2}\sin\frac{x-y}{2}\right| \leq 2\left|\sin\frac{x-y}{2}\right| \leq 2\left|\frac{x-y}{2}\right| < \varepsilon.$$

Choose $\delta = \varepsilon$.

Other cases should be similar.

Note: $f(x) = x$ is uniformly continuous. Functions dominated by this function should be uniformly continuous as well.

4a $\quad e^{x^2}\sin x^2 = f(x)$

Take $x_n = \sqrt{n\pi}$, $\quad y_n = \sqrt{n\pi + \frac{\pi}{2}}$

$$|x_n - y_n| = \left|\sqrt{n\pi} - \sqrt{n\pi+\frac{\pi}{2}}\right| = \frac{\left|n\pi - \left(n\pi + \frac{\pi}{2}\right)\right|}{\left|\sqrt{n\pi} + \sqrt{n\pi+\frac{\pi}{2}}\right|} = \frac{\frac{\pi}{2}}{\left|\sqrt{n\pi}+\sqrt{n\pi+\frac{\pi}{2}}\right|}$$

$\to 0$ as $n \to \infty$.

By $$\left|f(x_n) - f(y_n)\right| = \left|e^{n\pi}\sin n\pi - e^{n\pi+\frac{\pi}{2}}\sin\left(n\pi+\frac{\pi}{2}\right)\right| \not\to 0$$

(check).

$\therefore$ $f(x)$ is not uniformly continuous.

(10) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

$|x| < \frac{3}{10}$.

Let me take approximation $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$

Then $\frac{x^9}{9!} - \frac{x^{11}}{11!} + \cdots < 5 \times 10^{-4}$.

Check if this holds. If not go one step ahead and see. If this holds, check with $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!}$.

The approximation should consist of the minimal no. of terms such that the rest is less than 5×10^{-4}.

(13) (1) $\frac{1}{1+x}$

I did this result in class ; G P sum

$|x| < 1$ Then $\frac{1}{1+x} = \frac{1}{1-(-x)} = 1 + (-x) + (-x)^2 + \cdots$

$$= 1 - x + x^2 - x^3 + \cdots$$

$|x| > 1$ $\frac{1}{1+x} = \frac{1}{x\left(\frac{1}{x}+1\right)} = \frac{1}{x}\left(\frac{1}{1-\left(-\frac{1}{x}\right)}\right)$ $\left[\left|\frac{1}{x}\right| < 1\right]$.

$$= \frac{1}{x}\left(1 - \frac{1}{x} + \frac{1}{x^2} - \frac{1}{x^3} + \cdots\right)$$

(2) $\sinh x = \dfrac{e^x - e^{-x}}{2}$

$$\left.\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \\ e^{-x} &= 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots \end{aligned}\right\} \quad \frac{e^x - e^{-x}}{2} = \cdots$$

(3) $e^x \sinh x$ Multiply the taylor expansion of e^x with the taylor expansion of $\sinh x$.

(4) $x \sin x$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$x \sin x = x^2 - \frac{x^4}{3!} + \frac{x^6}{5!} - \frac{x^8}{7!} + \cdots$$

12. (a) $\displaystyle\sum_{n=0}^{\infty} (n+1+2^n)\, x^n$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+2+2^{n+1})}{(n+1+2^n)}$$

$$= \frac{n+2}{n+1+2^n} + \frac{2^{n+1}}{n+1+2^n} \to 2.$$

$$\frac{n+2}{n+1+2^n} \to 0 \qquad \frac{2^{n+1}}{n+1+2^n} \to 2$$

I'm not sure; please check if this is correct.

Radius of convergence is $\frac{1}{2}$.

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(b) $\sum_{n=0}^{\infty} \frac{x^{2n}}{a^n}$

$\limsup \left(\frac{1}{a^n}\right)^{1/n} = \limsup \left(\frac{1}{a}\right) = \frac{1}{a}$, bounded.

radius of convergence $= a$

(a) $ROC = \dfrac{1}{\limsup (n+1+2^n)^{1/n}}$

(c) $ROC = \dfrac{1}{\limsup\limits_{n\in\mathbb{N}} \frac{1}{(n!\,n^n)^{1/n}}} = \dfrac{1}{\limsup (n!)^{1/n} \cdot n} = 0$

Q5 Check differentiability using definition only at $x=0$

$\lim\limits_{x\to 0} \frac{f(x)-f(0)}{x}$ exists or not.

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$|f(x) - f(y)| \leq (x-y)^2$

f is differentiable.

So, $f'(x) = \lim\limits_{y\to x} \frac{|f(x)-f(y)|}{|x-y|} \leq \lim\limits_{y\to x} |x-y| = 0$

$f'(x)=0 \Rightarrow$ rate of change of value of $f(x)$ with changing x is zero, i.e., $f(x)$ is a constant function.

Q6 Evaluate f' normally and check continuity at doubtful points.