The Davenport Constant For Finite Abelian Groups And Its r-wise Generalization

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tcg crest
Inventing Harmonious Future

[based on joint work with Dr. Eshita Mazumdar]

Students' Talk
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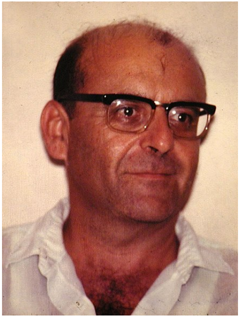
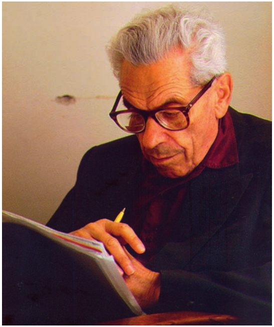
Theorem (Erdős, Ginzburg and Ziv, 1961)

Every $(2n-1)$-length sequence from C_n shall have a zero-sum subsequence of length n.

Problems





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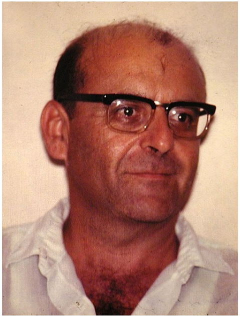

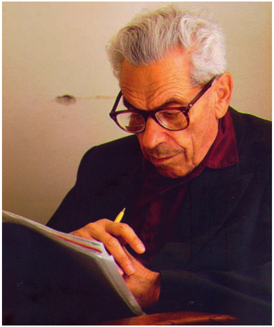

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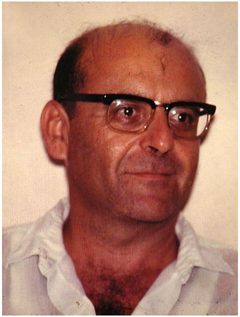

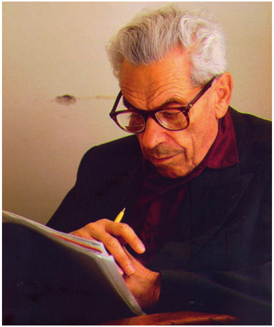

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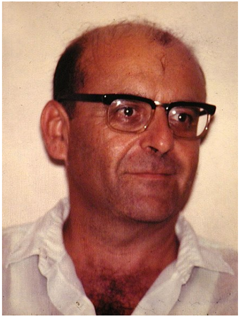

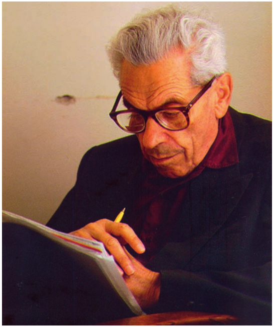

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An Upper Bound

Theorem

$D(G) \leq |G|$

Proof. Let $S = (x_1, x_2, \ldots, x_n) \in \mathcal{F}(G)$ where $|G| = n$. Consider

$$\begin{aligned} s_1 &= x_1 \\ s_2 &= x_1 + x_2 \\ &\vdots \\ s_n &= x_1 + x_2 + \cdots + x_n \end{aligned}$$

- All s_i's are distinct, so $0 \in \{s_1, \ldots, s_n\}$.
- By Pigeon-Hole Principle, $\exists\ i \neq j \ni s_i = s_j$. Then, $(x_{i+1}, \ldots, x_j)$ is a zero-sum subsequence.

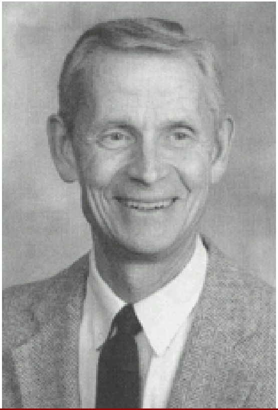
So, $D(G) \leq |G|$.

Breaking Up Into Cyclic Groups

Particularly for $G = C_n = \langle 1 \rangle$ (the cyclic group of order n) we can construct the sequence $S = \underbrace{11\ldots1}_{n-1}$ such that $0 \notin [S]$. Thus

$D(C_n) \geq n \Rightarrow D(C_n) = n$.

We denote by $[S]$ the set of subsums of sequence S including $\sigma(S)$.



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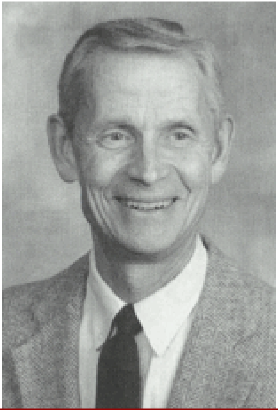

[Olson, 1969]

If $G \cong C_{n_1} \times C_{n_2}$, then $D(G) = n_1 + n_2 - 1$.

To prove this, Olson used a tool that'd become extremely useful in future.

Theorem

Let $G \cong H \times K$; $|H| = h, |K| = k$ and $h|k$. If S is a sequence over $G \ni \ |S| \geq h + k - 1$, then $0 \in [S]$.

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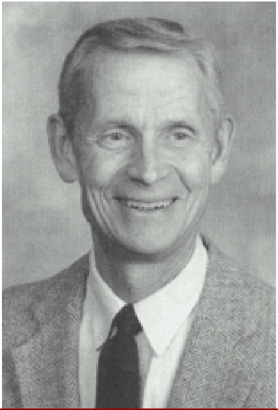

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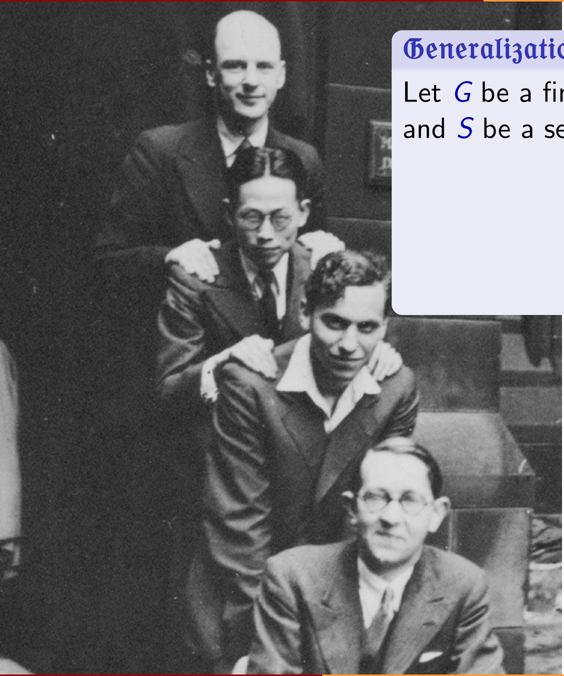
MATHEMATICS
DEPARTMENT



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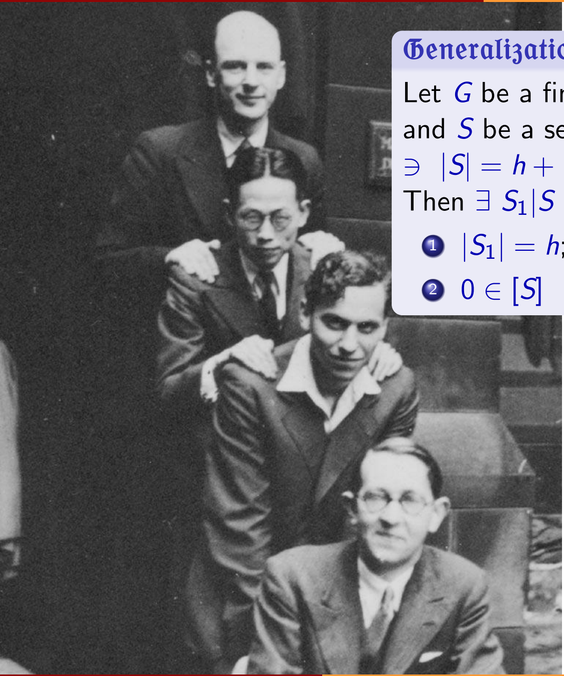

Generalization of EGZ theorem

Let G be a finite abelian group and S be a sequence over G $\ni$ $|S| = h + |G| - 1$ where $h \mid \text{order}(G)$. Then $\exists\ S_1 | S\ \ni$

1. $|S_1| = h$;
2. $0 \in [S]$

Embed G in the direct product $C_h \times G$. Let 1_h be a generator of C_h. Consider the sequence

$$(1_h, S) = \prod_{g \in S} (1_h, g)$$

Applying Theorem [**?**], we get $0 \in (1_h, S)$. But since $order(1_h) = h$, that zero-sum subsequence must be of length h.

p-Groups

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Is this conjecture true?

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For odd $m, n \ni 3 \leq m|n$,

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Yet it remains to be seen

1. for which groups Olson's conjecture holds
2. whether true for all groups of rank 3.

Groups Of Rank 3

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$S = (x_1, y_1) \dots (x_{4p-2}, y_{4p-2}) \in \mathcal{F}(G)\ \ni$

$$\begin{aligned} y_1 = \cdots = y_r &= 1 \\ y_{r+1} = \cdots = y_{4p-2} &= 0 \end{aligned}$$

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$D(C_p \oplus C_p \oplus C_{2p}) = D^*(C_p \oplus C_p \oplus C_{2p})\ \forall\ p \in \mathbb{P}$ for which the Conjecture holds.

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5. Delorme et al.

If $p \in \mathbb{P},\; 2 \leq n \in \mathbb{N}$ & $gcd(m, p^n) = 1$, then
$D(C_p \times C_p \times C_{p^n m} = D^*(C_p \times C_p \times C_{p^n m})$.

Summer 2022, we were looking at further generalizations of this.

Conjecture 2

Fix p, q $(p \neq q), p \in \mathbb{P}$; define $G := C_p^d \times C_q$.
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Looking back at the past, we shall present the results that were discovered for D_r almost in the same order as with D.

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- If $n, r \in \mathbb{N}$,

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- If $n, r \in \mathbb{N}$,
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Maximal r-wise zero-sum free sequence:
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- **What about higher ranks.**

for p-groups

Let p be an odd prime and $G \cong C_{p^{e_1}} \times C_{p^{e_2}} \times \cdots \times C_{p^{e_d}} \ni e_i \leq e_{i+1}$.

Maximal lower bound:

$$\underbrace{(1, 0, \ldots, 0)}_{d}{}^{rp^{e_d}-1}(0, 1, \ldots, 0)^{p^{e_{d-1}}-1} \ldots (0, 0, \ldots, 0, 1)^{p^{e_1}-1}.$$

$$\therefore D_r(G) \geq rp^{e_d} + \sum_{i=1}^{d-1} p^{e_i} - k + 1.$$

Theorem (Target to prove)

If $p^{e_d} \geq 1 + \sum_{i=1}^{d-1} (p^{e_i} - 1)$,

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$$D_r(G) = rp^{e_d} + \sum_{i=1}^{d-1} p^{e_i} - k + 1.$$

Consider a p-group $G = C_{p^{e_1}} \times C_{p^{e_2}} \times \cdots \times C_{p^{e_d}}$ with $1 \leq e_i \leq e_{i+1}$ for $i \in [1, d-1]$.

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$\eta_r(G) = \min\{k \in \mathbb{N} \mid \forall\ S \in \mathcal{F}(G) \ni |S| \geq k, 0^r_{\text{small}} \in \sigma(S)\}$.

By 0_{small} we mean a small zero subsum, i.e., a subsum of length $\leq \exp(G)$.

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Theorem (Fan, Gao, Wang, Zhong (2013))

$m \in \mathbb{N}$; let H be a finite abelian group $\ni$

- $exp(H)|m$
- $m \geq D(H)$
- $D(C_m \times C_m \times H) = 2m + D(H) - 2$

Then $\eta_1(C_m \times H) \leq 2m + D(H) - 2$.

Lemma 1

If $p^{e_d} \geq 1 + \sum_{i=1}^{d-1} (p^{e_i} - 1)$, then $\eta(G) \leq D(G) + \exp(G)$.

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The theorem for p-groups has nothing more to prove. A sketch can be found in the monograph by Geroldinger and Halter-Koch, where it is proved as a byproduct of algebraic ideas developed in the book. This approach, though in essence is the same, involves more elementary ideas and can be grasped by a first-year undergraduate who is attentive enough. The combinatorial approach provides, more than the proof, an intuitive idea in the structure of the group invariants.

Any better than for p-groups?

Theorem (Delorme, Ordaz and Quiroz (2001))

$$D\left(C_p \times C_p \times C_{p^n m}\right) = D^*\left(C_p \times C_p \times C_{p^n m}\right)$$

where p *is prime,* $n \geq 2,\ \gcd\left(m, p^n\right) = 1$.

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where p is prime, $n \geq 2$, $\gcd(m, p^n) = 1$.

We could get this far with exact values:

For $G = C_{p^{e_1}} \times C_{p^{e_2}} \times \cdots \times C_{p^{e_{d-1}}} \times C_{mp^{e_d}}$ with $e_i \leq e_{i+1}$ such that

$p^{e_d} \geq 1 + \sum_{i=1}^{d-1} (p^{e_i} - 1)$,

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1. increasing r
2. with larger primes $p_1, \ldots, p_\ell$
3. higher powers e_i^j $(j = 1, \ldots, \ell; i = 1, \ldots, d)$

$$\text{error} \to 0$$

My only justification is that this is not as pointless as it looks. In fact, the formula, though painful for the eye, gives quite nice output.

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$$\text{i.e., } \frac{\text{upper bound}}{\text{lower bound}} \to 1$$

.

[error has been throughout multiplied by 100 for easy visualization]

Group		$r=1$	$r=2$	$r=3$	$r=4$	$r=5$
$C_{2.3.5} \oplus C_{2^2.3^2.5^2} \oplus C_{2^3.3^3.5^3}$	UB	41778	68778	95778	122778	149778
	LB	27928	54928	81928	108928	135928
	diff	13850				
	err	49.59181	25.21483	16.90509	12.71482	10.18922
$C_{3.5.7} \oplus C_{3^2.5^2.7^2} \oplus C_{3^3.5^3.7^3}$	UB	1596033	2753658	3911283	5068908	6226533
	LB	1168753	2326378	3484003	4.641628	5799253
	diff	427280				
	err	36.55862	18.36675	12.26405	9.205391	7.367845
$C_{5.7.11} \oplus C_{5^2.7^2.11^2} \oplus C_{5^3.7^3.11^3}$	UB	69921553	126988178	184054803	241121428	298188053
	LB	57215233	114281858	171348483	228415108	285481733
	diff	12706320				
	err	22.207932	11.118405	7.415484	5.562819	4.450835

Group		$r=1$	$r=5$
$C_{31.47.101} \oplus C_{31^2.47^2.101^2} \oplus C_{31^3.47^3.101^3}$	UB	3292613286703417	16039460139018988
	LB	3186733368408697	15933580220724268
	diff	105879918294720	
	err	3.3225220	0.6645080

$C_{31^2.47^3.101^3} \oplus C_{31^8.47^9.101^5} \oplus C_{31^{17}.47^{21}.101^7}$

$r=1$

Upper bound=3143789277070391170765946419604722056999182463937961348553475489045413888OO

Lower bound= 3143789277070272156917043488137439730431718446582712047894248450951049379 84

Difference= 118904470778768425226721896032537391818273330798491684160143 36

Error= 0.000000000037822023

$r=5$

Upper bound= 1571894638535147728759342621748217544441028696720224309918829150723580624896

Lower bound=1571894638535135877591266211694935422470597680646030708671750002626420277248

Error= 0.000000000000756440

In fact, I can prove

Result

The error becomes negligible, i.e., $\frac{\text{upper bound}}{\text{lower bound}} \to 1$ if

- either p_j's are large;
- $e_i^{(j)}$'s are higher natural numbers;
- r increases.

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$$\begin{aligned}\text{difference} &= \text{upper bound} - \text{lower bound} \\ &= \sum_{m=1}^{\ell-1}\left(\left(\prod_{j=m+1}^{\ell} p_j^{e_d^{(j)}}\right)\varphi(p_m)\right) + \varphi(p_\ell) - \sum_{i=1}^{d-1}\left(\prod_{j=1}^{\ell} p_j^{e_i^{(j)}} - 1\right)\end{aligned}$$

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INDEPENDENT OF r

- For $d = 1$, these two bounds coincide.

- We are able to conclude about $D_r(G)$ for certain class of G with $\exp(G) = p^{e_d}$, where $e_d > 1$

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What if $\exp(G) = p$

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Conjecture 3

For prime p and $r, d \in \mathbb{N}$, $D_r((C_p)^d) = (r + d - 1)p - (d - 1)$.

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Let p, q be distinct primes and $G \cong C_p^{d-1} \times C_{pq}$ of rank $d \geq 3$.

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Relating the Conjectures

Let p, q be distinct primes and $G \cong C_p^{d-1} \times C_{pq}$ of rank $d \geq 3$. If the previous Conjecture holds for prime p, then $D(G) = D^*(G)$.

$D^*(G) = (r + d - 1)p - (d - 1)$ in this case.

Table of Contents

References: I

- Gautami Bhowmik and Jan-Christoph Schlage-Puchta, *Davenport's constant for groups with large exponent*, Contemporary Mathematics, Vol 579, (2012).
- Gautami Bhowmik and Jan-Christoph Schlage-Puchta, *Davenport's constant for groups of the form* $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_{3d}$. In Additive combinatorics, volume 43 of CRM Proc. Lecture Notes, pages 307-326. Amer. Math. Soc., Providence, RI, (2007).
- Charles Delorme, Oscar Ordaz and Domingo Quiroz, *Some remarks on Davenport constant*, Discrete Mathematics, 237 (2001), 119–128.
- Y.S. Fan, W.D. Gao, L.L. Wang and Q. H. Zhong, *Two zero-sum invariants on finite abelian groups*, European J. Combinatorics, 34 (2013), 1331–1337.
- W Gao, P Zhao, J Zhuang, *Zero-sum subsequences of distinct lengths*, Int. J. Number Theory, Vol 11 (7), (2015) 2141-2150.
- A. Geroldinger and F. Halter-Koch, *Non-Unique Factorizations*, Chapman & Hall/CRC (2006).
- A. Geroldinger, R. Schneider, On Davenport's constant, J. Combin. Theory Ser. A 61 (1992) 147–152.
- Benjamin Girard and Wolfgang A. Schmid, *Direct zero-sum problems for certain groups of rank three*, J. Number Theory 197 (2019) 297–316.
- F. Halter-Koch, Arithmetical interpretation of weighted Davenport constants, *Arch. Math.* **103** (2014), 125-131.
- John E. Olson, *A Combinatorial Problem on Finite Abelian Groups, I*, Journal of Number Theory, 1 (1969), 8–10.

References: II

John E. Olson, *A Combinatorial Problem on Finite Abelian Groups, II*, Journal of Number Theory, 1 (1969), 195–199.

K. Rogers, A Combinatorial problem in Abelian groups, *Proc. Cambridge Phil. Soc.* **59** (1963), 559-562.

A. Sheikh, The Davenport constant of finite abelian groups, *Thesis, University of London,* (2017).

Thank you :)