# Dots, shapes, homotopy

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Alumni Interaction Series
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# The seven bridges problem

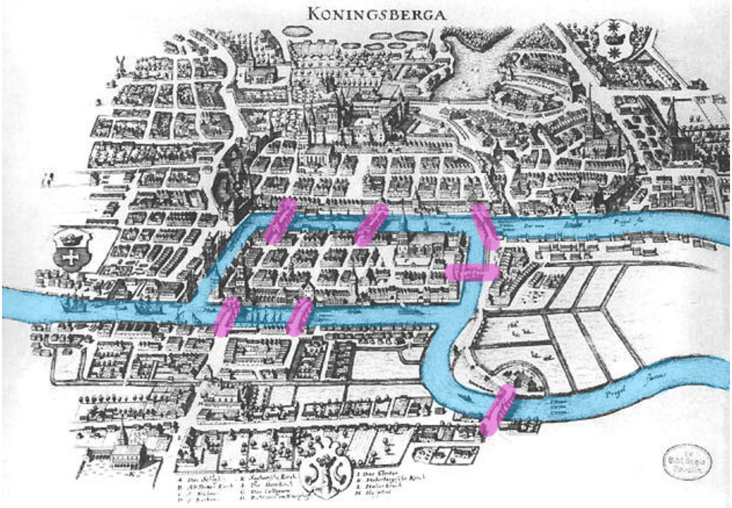
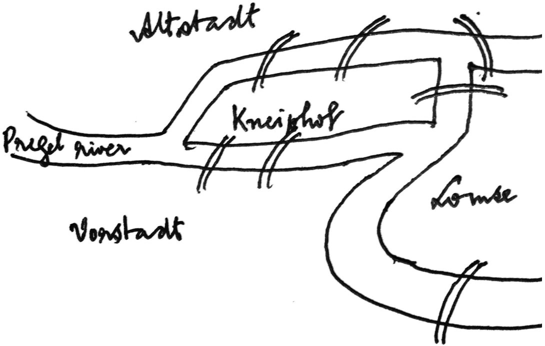
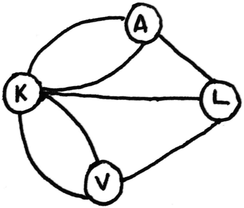



Figure: Königsberg

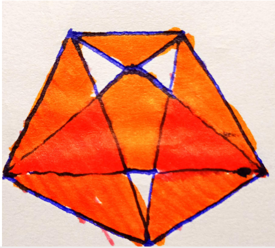
Problem: A walk through the city that would cross each bridge exactly once.



Three filled-in triangles, each pair intersect at an unique vertex.

Figure: 3 simplices touching and homotopy type



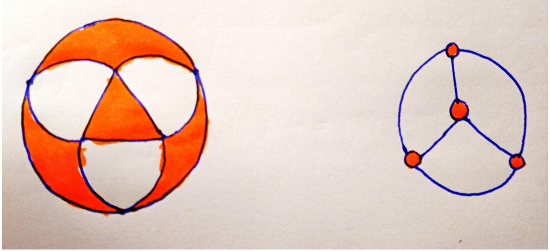
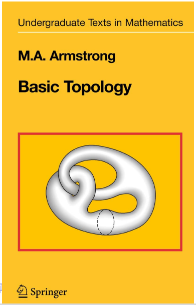


Figure: Transliteration into the Complete Graph

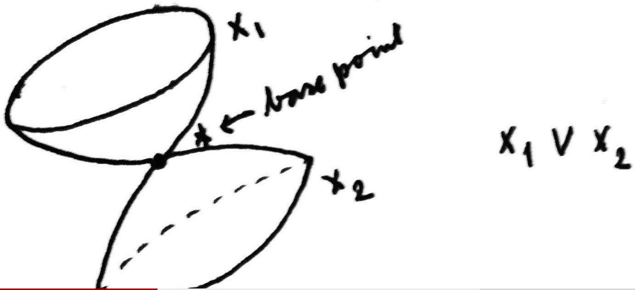
$K_m$, complete graph with $m$ vertices, every pair of vertices has a unique edge between them. No. of circles $(\mathbb{S}^1\text{'s})$ made in a simple graph is $\#\text{edges} - \#\text{vertices} + 1$. So, for $K_m$, it is $\binom{m-1}{2}$.



## Wedge

$(X_\alpha, x_\alpha)\,;\alpha \in A$, based spaces. Quotient the disjoint union $\coprod X_\alpha$ by $\{x_\alpha \mid \alpha \in A\}$ (identify the base points into one) for *wedge* of $\{(X_\alpha, x_\alpha)\}_{\alpha\in A}$, $\bigvee\limits_{\alpha\in A} X_\alpha$.

For each $\alpha \in A$, injection $i_\alpha : X_\alpha \to \bigvee\limits_{\alpha\in A} X_\alpha; x \mapsto [x] \ \forall\, x \in X_\alpha$. Maps $f_\alpha : X_\alpha \to Y$ determine unique $f : \bigvee\limits_{\alpha\in A} X_\alpha \to Y \ \ni \ fi_\alpha = f_\alpha$.

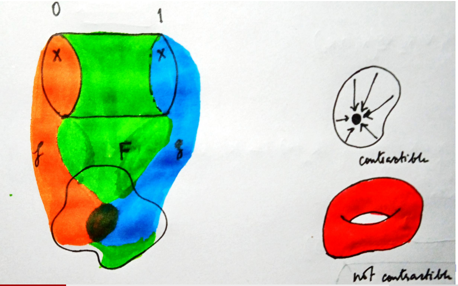


# Homotopy

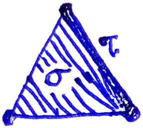
**Definition**

$f, g : X \to Y$, *maps* (continuous functions). Say $f \simeq g$ if $\exists$ $F : X \times I \to Y \ni F(x,0) = f(x), F(x,1) = g(x) \ \forall\, x \in X$. $F$ is a *homotopy* from $f$ to $g$; $F : f \simeq g$.

Suppose $\exists\ f : X \to Y, g : Y \to X \ni g \circ f \simeq \mathrm{id}_X, f \circ g \simeq \mathrm{id}_Y$. Then $X \simeq Y$, have the same homotopy type. A path-connected space is *contractible* if it has the homotopy type of a point.



# Simplicial complex




A *simplicial complex* $L$ on a finite set $V$ is a family of non-empty subsets of $V$ such that if $\sigma \in L$ and $\tau \subseteq \sigma$ is non-empty, then $\tau \in L$.

Čech complex: $\sigma$ is a simplex iff $\bigcap_{x \in \sigma} B\left(x, \frac{r}{2}\right) \neq \phi$.

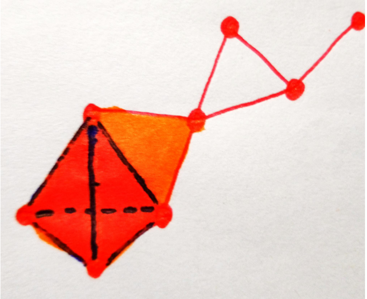


Figure: a typical simplicial complex

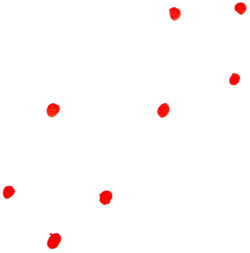


Figure: the vertex set

# Problem

$[m] = \{1, 2, \ldots, m\}$; $\mathcal{F}_n^m = \{A \subseteq [m] \ : \ |A| = n\}$.
Symmetric difference metric: $d\,(a, b) = |(a \setminus b) \cup (b \setminus a)|$

$$\check{\mathrm{C}}\mathrm{ech}\left(\mathcal{F}_n^m; r\right), \ \ r > 0$$

Ordering: $a, b \in [m]$

- $|a| > |b| \Rightarrow a \succ b$;
- When $|a| = |b|$...

Feng, Nukala: $\mathcal{VR}\left(\mathcal{F}_{\preceq a}^m\right)$
Alternative way to see: $a, b \in \mathbb{I}^m$.

- $|a - 0| > |b - 0| \Rightarrow a \succ b$;
- When $|a - 0| = |b - 0|$, reverse-lexicographic order.

## Open balls

For $v \in \mathcal{F}_n^m$,

- $b\left(v, \frac{r}{2}\right)$ $(0 \leq r < 4) = \{v\}$; hence each simplex is of the form $\sigma = \{\{v\}\}$ for $V \in \mathcal{F}_n^m$.
- For $4 \leq r < 8$, each simplex is of the form

  $$N^{(1)}[V] = \{a \in \mathcal{F}_n^m \ : \ \exists \ \{i_1, i_2, \ldots, i_{n-1}\} \subset v \ni \{i_1, i_2, \ldots i_{n-1}\} \subset a\},$$

  where $v \in \mathcal{F}_n^m$.
- For $8 \leq r < 12$, each simplex is of the form

  $$N^{(2)}[v] = \{a \in \mathcal{F}_n^m \ : \ \exists \ \{i_1, i_2, \ldots, i_{n-2}\} \subset V \ni \{i_1, i_2, \ldots i_{n-2}\} \subset a\}.$$

$\vdots$

- $B(v, r \geq 2n) = \mathcal{F}_n^m$. the whole space is a $(m^n - 1)$-simplex.

For $4k \leq r < 4\left(k+1\right)$, each simplex is of the form

$$N^{(k)}\left[v\right] = \left\{a \in \mathcal{F}_n^m \ : \ \exists \ \left\{i_1, i_2, \ldots, i_{n-k}\right\} \subset v \ni \left\{i_1, i_2, \ldots i_{n-k}\right\} \subset a\right\},$$

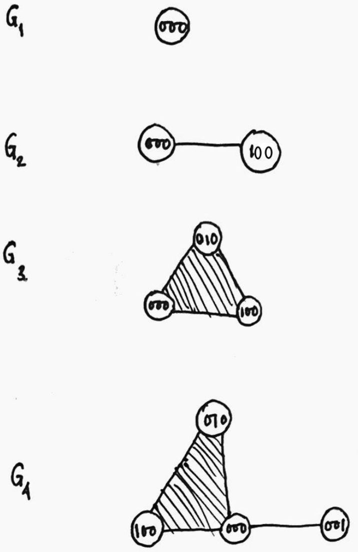
where $i_j \in [m]$ and $i_{j_1} \neq i_{j_2}$.

**Distances**

- $d\left(\left\{x, y, z\right\}, \left\{x, y, z\right\}\right) = 0$;
- $d\left(\left\{x, y, z\right\}, \left\{x, y, w\right\}\right) = 2$;
- $d\left(\left\{x, y, z\right\}, \left\{x, u, v\right\}\right) = 4$;
- $d\left(\left\{x, y, z\right\}, \left\{u.v.w\right\}\right) = 6$.

Only one coordinate differs for distance 2.

# Čech $\left(G_k, \frac{r}{2} = 1\right)$



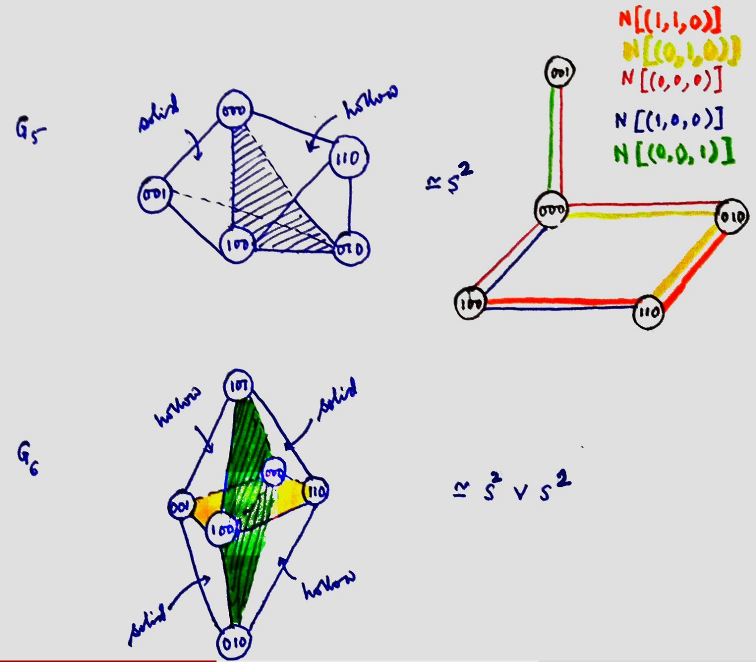

$|a| = 0 \to (0,0,0)$

lexicographic.

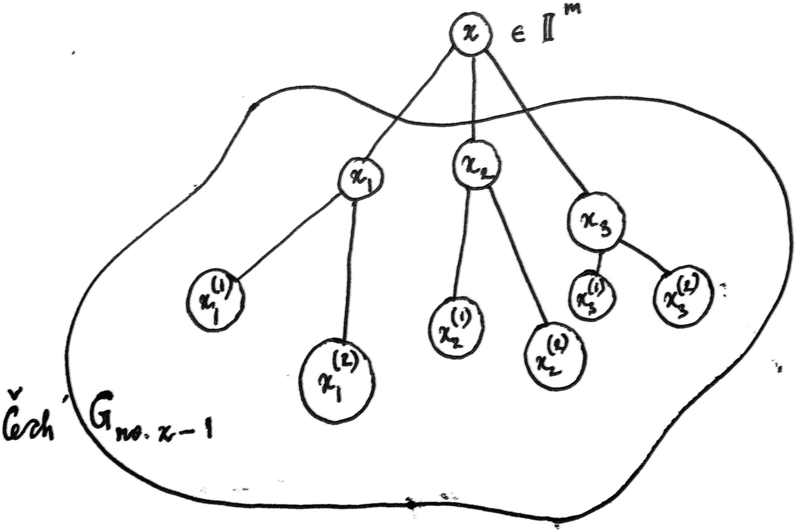
$|a| = 1 \to (0,0,1)$

$(0,1,0)$

$(1,0,0)$


$G_5$
solid
hollow
000
110
001
100
010
$\simeq S^2$
N[(1,1,0)]
N[(0,1,0)]
N[(0,0,0)]
N[(1,0,0)]
N[(0,0,1)]
$G_6$
101
solid
hollow
001
000
110
100
010
hollow
solid
$\simeq S^2 \vee S^2$

# Method



# Link-deletion

For any vertex $v$ in a complex $K$, *deletion* $K \setminus v$ denotes the induced complex on the vertex set $K^{(0)} \setminus \{v\}$. The *link* of $v$ in $K$, $\mathrm{lk}_K(v) = \{\sigma : \sigma \cup \{v\} \in K \text{ and } v \notin \sigma\}$.

**Lemma (Goyal, Shukla, Singh)**

*If $K$ is a simplicial complex and for $K = K_1 \cup K_2$, the inclusion maps $i_j : K_1 \cap K_2 \hookrightarrow K_j$ are both null-homotopic. Then $K \simeq K_1 \vee K_2 \vee \Sigma(K_1 \cap K_2)$.*

**Corollary (Adamaszek, Adams)**

*Let $v \in K$ be a vertex and the inclusion $\mathrm{lk}_K(v) \hookrightarrow K$ is null-homotopic. Then, $K \simeq K \setminus v \vee \Sigma(\mathrm{lk}_K(v))$.*

$\mathrm{lk}_{\check{\mathrm{C}}\mathrm{ech(no.}\ x)}(x)$: $(n+1)$ new facets:

- $N[x] = \langle x_1, x_2, \ldots, x_n \rangle$;
- $N[x_1] = \left\langle x_1, x_1^{(1)}, x_1^{(2)}, \ldots, x_1^{(n-1)} \right\rangle$;
- $N[x_2] = \left\langle x_2, x_2^{(1)}, x_2^{(2)}, \ldots, x_2^{(n-1)} \right\rangle$;

  $\vdots$
- $N[x_n] = \left\langle x_n, x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(n-1)} \right\rangle$,

where $N[x]$ is the simplex around $x$, $x_i$ is the face of $x$ with the $i$th non-zero-coordinate made 0, $x_i^{(j)}$ is the face of $x_i$ with the $j$th non-zero-coordinate made 0. By the Nerve theorem, since all intersections are contractible, $\mathrm{lk}_{\check{\mathrm{C}}\mathrm{ech}(G_{\mathrm{no.}\ x})}(x) \simeq K_{n+1} \simeq \bigvee_{\binom{n}{2}} \mathbb{S}^1$.

**Theorem (Nerve theorem)**

*Let $F$ be a finite collection of closed, convex sets in the Euclidean space. Then the nerve of $F$ and the union of the sets in $F$ have the same homotopy type.*

$$\mathrm{lk}_{\check{C}\mathrm{ech}(G_{\mathrm{no}.x})}(x) \simeq \bigvee_{\binom{n}{2}} S^1 \xmapsto{\Sigma} \bigvee_{\binom{n}{2}} S^2$$

CRL

$$x$$

$$\mathrm{del}_{\check{C}\mathrm{ech}(G_{\mathrm{no}.x})}(x) = \check{C}\mathrm{ech}(G_{\mathrm{no}.x-1})$$

RECURRENCE RELATION

# Result

For any $m, n \in \mathbb{N}$ and $a \in \mathcal{F}_n^m$,

$$\text{Čech}\left(\mathcal{F}^m_{\preccurlyeq a}\right) = \bigvee_{\sum_{i=2}^{n-1}\binom{m}{i}\binom{i}{2} + \#\{b \in \mathcal{F}_n^m \mid b \preccurlyeq a\}} \mathbb{S}^2.$$

If we take $a$ to be the last element (with CRL ordering) of $\mathcal{F}_n^m$, $a'$ of $\mathcal{F}_{n-1}^m$, then we get $\text{Čech}\left(\mathcal{F}^m_{\preccurlyeq a}\right) \setminus \text{Čech}\left(\mathcal{F}^m_{\preccurlyeq a'}\right) = \text{Čech}\left(\mathcal{F}_n^m\right)$.

## Conjecture

$$\text{Čech}\left(\mathcal{F}_3^m; 4 \leq r < 8\right) \simeq \left(\bigvee_{\gamma_m} \mathbb{S}^3\right) \vee \left(\bigvee_{(m-4)\alpha_m} \mathbb{S}^4\right)$$

where $\alpha_m = \binom{m}{4}$, $\gamma_m = (m-4)\,\alpha_m - \frac{3m+1}{4}\binom{m-2}{3}$. Jonsson showed it would be wedge of an unknown no. of $\mathbb{S}^3$'s and $\mathbb{S}^4$'s.

# References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Thank you :-)

https://anamitro.github.io/files/talks/nits_alumni_26.pdf