LEBESGUE MEASURE

Set fn. a function which associates an extended real no. to each set in a colln of sets.

σ-algebra. $\{A_\lambda\}_{\lambda \in \Lambda} = \mathcal{A}$.

(i) $\mathbb{R} \in \mathcal{A}$.

(ii) $A_\lambda \subseteq \mathbb{R} \ \forall \ A_\lambda \in \mathcal{A}$

(iii) $A_\lambda \in \mathcal{A} \Rightarrow A_\lambda^c \in \mathcal{A}$

(iv) $\bigcup_{i=1}^{\infty} A_{\lambda_i} \in \mathcal{A}$ if $A_{\lambda_i} \in \mathcal{A} \ \forall \ i \in \mathbb{N}$.

m : Lebesgue measure

(i) $\phi \neq I$, interval $\Rightarrow m(I) = \ell(I)$

(ii) $E+y = \{x+y \mid x \in E\}$ $\quad m(E+y) = m(E)$

(iii) $\{E_k\}_{k=1}^{\infty}$; $E_i \cap E_j = \phi$ for $i \neq j$ then $m\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} m(E_k)$.

Outer measure, m^*

countably subadditive.

$\{E_k\}_{k=1}^{\infty}$ $\quad m^*\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} m^*(E_k)$.

$$m^*(A) = \inf\left\{\sum_{k=1}^{\infty} \ell(I_k) \;\middle|\; A \subseteq \bigcup_{k=1}^{\infty} I_k\right\}$$

- $m^*(\phi) = 0$
- monotone $\quad A \subseteq B \Rightarrow m^*(A) \leq m^*(B)$.

Ex. A countable set has measure 0.

$C = \{c_k\}_{k=1}^{\infty}$. $\quad \varepsilon > 0$. $\forall \ k \in \mathbb{N}$, $I_k = \left(c - \frac{\varepsilon}{2^{k+1}}, c + \frac{\varepsilon}{2^{k+1}}\right)$. $C \subseteq \bigcup_{k=1}^{\infty} I_k$.

$\therefore 0 \leq m^*(C) \leq \sum_{k=1}^{\infty} \ell(I_k) = \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon$. Holds $\forall \ \varepsilon > 0 \Rightarrow m^*(E) = 0$.

Prop 1. Outer measure of an interval is its length.

Pf. Closed bounded interval $[a,b]$

$\varepsilon > 0$. $[a,b] \subset (a-\varepsilon, b+\varepsilon)$

$\Rightarrow m^*([a,b]) \leq \ell((a-\varepsilon, b+\varepsilon)) = b-a+2\varepsilon$.

holds $\forall \varepsilon > 0$. $\therefore m^*([a,b]) \leq b-a$.

• T.S.T : $m^*([a,b]) \geq b-a$.

$\$ \forall \{I_k^{\text{open, bdd}}\}_{k=1}^{\infty} \ni [a,b] \subseteq \bigcup_{k=1}^{\infty} I_k$, $\sum_{k=1}^{n} \ell(I_k) \geq b-a$.

HBT $\hookrightarrow \exists \{I_k\}_{k=1}^{n} \subset \{I_k\}_{k=1}^{\infty} \ni [a,b] \subseteq \bigcup_{k=1}^{n} I_k$.

~~Choose $n \in \mathbb{N}$~~ $\Rightarrow a \in (a_1, b_1)$

$\Rightarrow b \geq b_1 \Rightarrow \sum_{k=1}^{n} \ell(I_k) \geq b_1 - a_1 > b-a$

$\Rightarrow b_1 \in [a,b)$, $b_1 \notin (a_1,b_1)$, $\exists (a_2,b_2) \in \{I_k\}_{k=1}^{n} \ni b_1 \in (a_2,b_2)$
$ \neq (a_1,b_1)$

$\Rightarrow b_2 \geq b$, $\sum_{k=1}^{n} \ell(I_k) \geq (b_1-a_1)+(b_2-a_2) = b_2-(a_2-b_1)-a_1 > b_2-a_1 > b-a$

$\Rightarrow \{(a_k,b_k)\}_{k=1}^{N} \subseteq \{I_k\}_{k=1}^{n} \ni a_1 < a$ and $a_{k+1} < b_k$ for $1 \leq k \leq N-1$, and $b_N > b$.

Thus, $\sum_{k=1}^{n} \ell(I_k) \geq \sum_{k=1}^{N} \ell((a_i, b_i))$

$= (b_N - a_N) + (b_{N-1} - a_{N-1}) + \dots + (b_1 - a_1)$

$= b_N - (a_N - b_{N-1}) - \dots - (a_2 - b_1) - a_1$

$> b_N - a_1 > b-a$.

Any bounded interval. Given $\varepsilon > 0$, $\exists J_1^{\text{closed, bdd}}, J_2^{\text{closed, bdd}} \ni J_1 \subseteq I \subseteq J_2$

while $\ell(I) - \varepsilon < \ell(J_1)$ and $\ell(J_2) < \ell(I) + \varepsilon$.

By (i) equality of outer measure and length for closed, bdd intervals

(ii) monotonicity of outer measure

$$\ell(I) - \varepsilon < \ell(J_1) = m^*(J_1) \leq m^*(I) \leq m^*(J_2) = \ell(J_2) < \ell(I) + \varepsilon$$

holds $\forall\, \varepsilon > 0$. $\therefore\ \ell(I) = m^*(I)$.

<u>Unbounded interval</u> $\forall\, n \in \mathbb{N},\ \exists\, J \subseteq I \ni \ell(J) = n$

$\therefore\ m^*(I) \geq m^*(J) = \ell(J) = n$. holds $\forall\, n \in \mathbb{N}$. $\therefore\ m^*(I) = \infty$.

<u>Prop 2.</u> A, y $\quad m^*(A+y) = m^*(A)$

<u>Pf.</u> $\{I_k\}_{k=1}^{\infty}$; $A \subseteq \bigcup_{k=1}^{\infty} I_k$ iff $A + y \subseteq \bigcup_{k=1}^{\infty} (I_k + y)$

I_k open $\Rightarrow (I_k + y)$ open ; $\ell(I_k) = \ell(I_k + y) \Rightarrow \sum_{k=1}^{\infty} \ell(I_k) = \sum_{k=1}^{\infty} \ell(I_k + y)$.

<u>Prop 3.</u> $\{E_k\}_{k=1}^{\infty}$; $m^*\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} m^*(E_k)$

<u>Pf.</u> ~~If~~ $\exists\, K \in \mathbb{N} \ni m^*(E_K) = \infty \quad \to ✓$

Suppose $m^*(E_k) < \infty\ \forall\ k \in \mathbb{N}$.

<u>$\varepsilon > 0$</u>. $\forall\, k \in \mathbb{N}\ \exists\ \{I_{k,i}\}_{i=1}^{\infty}$ open, bdd countable $\ni E_k \subseteq \bigcup_{i=1}^{\infty} I_{k,i}$

$$\sum_{i=1}^{\infty} \ell(I_{k,i}) < m^*(E_k) + \frac{\varepsilon}{2^k}.$$

$\{I_{k,i}\}_{1 \leq k, i \leq \infty}$ countable

$$m^*\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{1 \leq k, i \leq \infty} \ell(I_{k,i}) = \sum_{k=1}^{\infty}\left(\sum_{i=1}^{\infty} \ell(I_{k,i})\right) < \sum_{k=1}^{\infty}\left(m^*(E_k) + \frac{\varepsilon}{2^k}\right) = \left(\sum_{k=1}^{\infty} m^*(E_k)\right) + \varepsilon.$$

holds $\forall\ \varepsilon > 0$, ∎

- $E_k = \phi$ for $k > n$. finite sub-additivity | $\{E_k\}_{k=1}^{n}$ $\quad m^*\left(\bigcup_{k=1}^{n} E_k\right) \leq \sum_{k=1}^{n} m^*(E_k)$.

$\exists$ sets $A, B \ni A \cap B = \phi$ & $m^*(A \cup B) < m^*(A) + m^*(B)$.

(Constantine Carathéodory)

Defn. E measurable $\Leftrightarrow$ $m^*(A) = m^*(A \cap E) + m^*(A \cap E^c)$ $\forall$ set A.

- A, measurable, B any set $\ni B \cap A = \phi$

$m^*(A \cup B) = m^*((A \cup B) \cap A) + m^*((A \cup B) \cap A^c) = m^*(A) + m^*(B)$.

- Prop 3 $\Rightarrow$ outer measure finitely subadditive ; $A = (A \cap E) \cup (A \cap E^c)$

  $\therefore m^*(A) \leq m^*(A \cap E) + m^*(A \cap E^c)$

  $\therefore E$ measurable iff $m^*(A) \geq m^*(A \cap E) + m^*(A \cap E^c)$.

- Trivial if $m^*(A) = \infty$.

- symmetric in E, E^c ; E measurable $\Leftrightarrow$ E^c is measurable.

  $\mathbb{R}, \phi$ measurable

- Prop 4. $m^*(E) = 0 \Rightarrow E$ mble.

Pf. A, any set

$A \cap E \subseteq E$ & $A \cap E^c \subseteq A$

monotonicity $\Rightarrow$ $m^*(A \cap E) \leq m^*(E) = 0$, $m^*(A \cap E^c) \leq m^*(A)$.

$\therefore m^*(A) \geq m^*(A \cap E^c) = 0 + m^*(A \cap E^c) = m^*(A \cap E) + m^*(A \cap E^c) \Rightarrow E$ mble. $\square$

Prop 5. Union of finite colln of mble sets is mble.

Proof. $E_1, E_2 \in \mathcal{M}$

$$m^*(A) = m^*(A\cap E_1) + m^*(A\cap E_1^c)$$

$$= m^*(A\cap E_1) + m^*((A\cap E_1^c)\cap E_2) + m^*((A\cap E_1^c)\cap E_2^c)$$

$$= m^*(A\cap E_1) + m^*((A\cap E_1^c)\cap E_2) + m^*(A\cap (E_1\cup E_2)^c)$$

$$\geq m^*(A\cap (E_1\cup (E_1^c\cap E_2))) + m^*(A\cap (E_1\cup E_2)^c) \quad \text{[by finite subadditivity]}$$

$$= m^*(A\cap (E_1\cup E_2)) + m^*(A\cap (E_1\cup E_2))^c$$

$\Rightarrow E_1\cup E_2 \in \mathcal{M}$.

$\boxed{\{E_k\}_{k=1}^n}$ Induction on n ; $\bigcup_{k=1}^{n} E_k = \left(\bigcup_{k=1}^{n-1} E_k\right)\cup E_n$. □

<u>Prop 6.</u> A, any set. $\{E_k\}_{k=1}^n \subset \mathcal{M}$; $E_i\cap E_j = \phi$ for $i\neq j$.

Then, $m^*\left(A\cap\left[\bigcup_{k=1}^n E_k\right]\right) = \sum_{k=1}^n m^*(A\cap E_k)$.

In particular,
$$m^*\left(\bigcup_{k=1}^n E_k\right) = \sum_{k=1}^n m^*(E_k).$$

<u>Proof.</u> Induction on $n\in\mathbb{N}$.

$\{E_k\}_{k=1}^n$ disjoint $\Rightarrow$ $A\cap\left[\bigcup_{k=1}^n E_k\right]\cap E_n = A\cap E_n$; $A\cap\left[\bigcup_{k=1}^n E_k\right]\cap E_n^c = A\cap\left[\bigcup_{k=1}^{n-1} E_k\right]$.

Measurability of E_n and induction assumption,

$$m^*\left(A\cap\left[\bigcup_{k=1}^n E_k\right]\right) = m^*(A\cap E_n) + m^*\left(A\cap\left[\bigcup_{k=1}^{n-1} E_k\right]\right)$$

$$= m^*(A\cap E_n) + \sum_{k=1}^{n-1} m^*(A\cap E_k)$$

$$= \sum_{k=1}^{n} m^*(A\cap E_k). \qquad □$$

- $\mathcal{A} = \{A_\lambda\}_{\lambda \in \Lambda}$ where $A_\lambda \subseteq \mathbb{R}\ \forall\ \lambda \in \Lambda$

$\downarrow$

<u>algebra</u> if $A_\lambda^c \in \mathcal{A}$ whenever $A_\lambda \in \mathcal{A}$

and $\bigcup_{k=1}^{n} A_{\lambda_k} \in \mathcal{A}\ \ \forall\ A_{\lambda_k} \in \mathcal{A}\ (k = 1, \dots, n)\ \forall\ n \in \mathbb{N}$.

DeMorgan $\Rightarrow \bigcap_{k=1}^{n} A_{\lambda_k} \in \mathcal{A}\ \ \forall\ A_{\lambda_k} \in A\ (k = 1, \dots, n)\ \forall\ n \in \mathbb{N}$.

- Union of a countable collection of measurable sets is also the union of a countable disjoint colln of measurable sets.

$\hookrightarrow\ \{A_k\}_{k=1}^{\infty}$

$A_1' = A_1 \quad \forall\ k \geq 2,\ A_k' := A_k \sim \bigcup_{i=1}^{k-1} A_i$.

$\mathcal{M}$, algebra $\Rightarrow A_i' \in \mathcal{M}$ ~~countable~~ $\forall\ i \in \mathbb{N}$

$$\bigcup_{k=1}^{\infty} A_k = \bigcup_{k=1}^{\infty} A_k'.$$

<u>$\mathcal{M}$, σ-algebra</u>

<u>Prop 7.</u> $E = \bigcup_{k=1}^{\infty} E_k$

E_k ~~msble~~ $\in \mathcal{M}\ \ \forall\ k \in \mathbb{N} \Rightarrow E \in \mathcal{M}$.

<u>Pf.</u> A, any set.

$n \in \mathbb{N}$. $\quad F_n = \bigcup_{k=1}^{n} E_k \quad ; \quad F_n \in \mathcal{M}$ & $F_n^c \supseteq E^c$

$m^*(A) = m^*(A \cap F_n) + m^*(A \cap F_n^c) \geq m^*(A \cap F_n) + m^*(A \cap E^c)$ $\quad [\because F_n \subseteq E \Rightarrow F_n^c \supseteq E^c \Rightarrow A \cap F_n^c \supseteq A \cap E^c]$

Prop 6 $\Rightarrow m^*(A \cap F_n) = \sum_{k=1}^{n} m^*(A \cap E_k)$.

Thus,

$$m^*(A) \geq \sum_{k=1}^{n} m^*(A\cap E_k) + m^*(A\cap E^c).$$

independent of n $\Rightarrow$ $m^*(A) \geq \sum_{k=1}^{\infty} m^*(A\cap E_k) + m^*(A\cap E^c)$

countable subadditivity $\geq m^*(A\cap E) + m^*(A\cap E^c)$

<u>Prop 8.</u> $a \in \mathbb{R}$; $(a,\infty) \in \mathcal{M}$.

<u>Pf.</u> A, any set.

$a \notin A$.

[$a\in A$, then replace A by $A\sim\{a\}$; $m^*(A) = m^*(A\sim\{a\})$]

$A_1 = A\cap(-\infty, a)$; $A_2 = A\cap(a,\infty)$.

<u>T.S.T.:</u> $m^*(A_1) + m^*(A_2) \leq m^*(A)$.

defn by infimum.

Let $\{I_k \text{ open, bdd}\}_{k=1}^{\infty}$; $A \subseteq \bigcup_{k=1}^{\infty} I_k$.

intervals

<u>T.S.T.:</u> $m^*(A_1) + m^*(A_2) \leq \sum_{k=1}^{\infty} \ell(I_k)$.

$I_k' = I_k \cap (-\infty, a)$; $I_k'' = I_k\cap(a,\infty)$ $\Rightarrow \ell(I_k) = \ell(I_k') + \ell(I_k'')$.

$\because \{I_k' \text{ open, bdd}\}_{k=1}^{\infty}$; $A_1 \subseteq \bigcup_{k=1}^{\infty} I_k'$

$\therefore$ by defn of outer measure, $m^*(A_1) \leq \sum_{k=1}^{\infty} \ell(I_k')$. Similarly, $m^*(A_2) \leq \sum_{k=1}^{\infty} \ell(I_k'')$.

$$m^*(A_1) + m^*(A_2) \leq \sum_{k=1}^{\infty} \ell(I_k') + \sum_{k=1}^{\infty} \ell(I_k'') = \sum_{k=1}^{\infty} \left[\ell(I_k') + \ell(I_k'')\right] = \sum_{k=1}^{\infty} \ell(I_k). \quad \square$$

- every open set, disjoint union of countable colln of open intervals

G_δ set intersection of a countable colln of open sets

F_σ set union of a countable colln of closed sets

Borel σ-algebra. The intersection of all σ-algebras of $\mathbb{R}$ that contain the open sets is a σ-algebra called the Borel σ-algebra; members of this colln are called Borel sets.

Thm 9. The collection $\mathcal{M}$ of measurable sets is a σ-algebra that contains σ-algebra $\mathcal{B}$ of Borel sets. Each interval, each open set, each closed set, each G_δ set, and each F_σ set is measurable.

Prop 10. Translate of a measurable set is measurable.

Pf. $E \in \mathcal{M}$

A, any set; $y \in \mathbb{R}$

$$m^*(A) = m^*(A-y) = m^*([A-y] \cap E) + m^*([A-y] \cap E^c)$$
$$= m^*(A \cap [E+y]) + m^*(A \cap [E+y]^c). \quad \square$$

Excision property. $A \in \mathcal{M}$, $m^*(A) < \infty$; $A \subseteq B$. Then

$$m^*(B) = m^*(B \cap A) + m^*(B \cap A^c) = m^*(A) + m^*(B \sim A)$$

$$\Rightarrow m^*(B \sim A) = m^*(B) - m^*(A) \qquad [\because m^*(A) < \infty].$$

Thm 11. $E \subseteq \mathbb{R}$. TFAE :

(o) $E \in \mathcal{M}$

(Outer approximation by open sets and G_δ sets)

(i) $\forall\ \varepsilon>0,\ \exists\ \mathcal{O}^{\text{open}} \ni E \subseteq \mathcal{O}$ & $m^*(\mathcal{O} \sim E) < \varepsilon$.

(ii) $\exists\ G_\delta$-set $G \ni E \subseteq G$ & $m^*(G \sim E) = 0$.

(Inner approximation by closed sets and F_σ sets)

(iii) $\forall\ \varepsilon>0,\ \exists\ F^{\text{closed}} \ni E \subseteq F$ & $m^*(E \sim F) < \varepsilon$.

(iv) $\exists\ F_\sigma$-set $F \ni E \subseteq F$ & $m^*(E \sim F) = 0$.

Pf. (Obs.) $E \in \mathcal{M} \Leftrightarrow E^c \in \mathcal{M}$

E open $\Leftrightarrow E^c$ closed

$E,\ F_\sigma \Leftrightarrow E^c,\ G_\delta$

(0) $\Rightarrow$ (i) $\quad E \in \mathcal{M}$. Let $\varepsilon > 0$.

Case 1. ($m^*(E) < \infty$). $\exists\ \{I_k^{\text{open interval}}\}_{k=1}^{\infty} \ni E \subseteq \bigcup_{k=1}^{\infty} I_k$ & $\sum_{k=1}^{\infty} \ell(I_k) < m^*(E) + \varepsilon$.

$\mathcal{O} := \bigcup_{k=1}^{\infty} I_k$. $E \subseteq \mathcal{O}$; $\quad m^*(\mathcal{O}) \leq \sum_{k=1}^{\infty} \ell(I_k) < m^*(E) + \varepsilon \Rightarrow m^*(\mathcal{O}) - m^*(E) < \varepsilon$.

But $E \in \mathcal{M}$, $m^*(E) < \infty$. By Excision property of $\mathcal{M}$, $m^*(\mathcal{O} \sim E) = m^*(\mathcal{O}) - m^*(E) < \varepsilon$.

Case 2. ($m^*(E) = \infty$). $\exists\ \{E_k\}_{k=1}^{\infty} \subseteq \mathcal{M} \ni m^*(E_k) < \infty\ \forall\ k \in \mathbb{N}$, $E = \bigcup_{k=1}^{\infty} E_k$

By case 1, $\forall\ k \in \mathbb{N}$, $\exists\ \mathcal{O}_k^{\text{open}} \ni E_k \subseteq \mathcal{O}_k$ & $m^*(\mathcal{O}_k - E_k) < \frac{\varepsilon}{2^k}$.

$\mathcal{O} = \bigcup_{k=1}^{\infty} \mathcal{O}_k$ open, $E \subseteq \mathcal{O}$ and $\mathcal{O} \sim E = \bigcup_{k=1}^{\infty} \mathcal{O}_k \sim E \subseteq \bigcup_{k=1}^{\infty} [\mathcal{O}_k - E_k]$.

$\therefore\ m^*(\mathcal{O} - E) \leq \sum_{k=1}^{\infty} m^*(\mathcal{O}_k \sim E_k) < \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon$.

(i) $\Rightarrow$ (ii) $\forall\ k \in \mathbb{N}$, choose $\mathcal{O}_k^{\text{open}} \ni E \subseteq \mathcal{O}_k$ and $m^*(\mathcal{O}_k \sim E) < \frac{1}{k}$. $\quad G := \bigcap_{k=1}^{\infty} \mathcal{O}_k$.

G, G_δ-set ; $E \subseteq G$.

$\because \forall k \in \mathbb{N}, \ G \sim E \subseteq \mathcal{O}_k \sim E,$

$\therefore$ monotonicity $\Rightarrow m^*(G \sim E) \leq m^*(\mathcal{O}_k \sim E) < \frac{1}{k} \Rightarrow m^*(G \sim E) = 0.$

<u>(ii) $\Rightarrow$ (0)</u> $m^*(G \sim E) = 0 \Rightarrow G \sim E \in \mathcal{M}$; $G \in \mathcal{M}$ being a G_δ-set ; $\mathcal{M}$ algebra

$\therefore E = G \cap (G \sim E)^c \in \mathcal{M}.$ $\square$

<u>Thm 12.</u> $E \in \mathcal{M}$, $m^*(E) < \infty$. $\forall \varepsilon > 0 \ \exists \{I_k^{\text{open interval}}\}_{k=1}^{\infty} \ni$ if $\mathcal{O} = \bigcup_{k=1}^{n} I_k$, then

$$m^*(E \Delta \mathcal{O}) = m^*([E \sim \mathcal{O}] \cup [\mathcal{O} \sim E]) \leq m^*(E \sim \mathcal{O}) + m^*(\mathcal{O} \sim E) < \varepsilon.$$

<u>Pf.</u> <u>Thm 11 (i)</u> $\Rightarrow \exists \mathcal{U}^{\text{open}} \ni E \subseteq \mathcal{U}, \ m^*(\mathcal{U} \sim E) < \frac{\varepsilon}{2}$.

$\because E \in \mathcal{M}$ and $m^*(E) < \infty$, excision $\Rightarrow m^*(\mathcal{U}) < \infty$.

• Every open set of real nos. is the disjoint union of a countable colln of open intervals.

$$\mathcal{U} = \bigcup_{k=1}^{\infty} I_k.$$

$$m^*(I_k) = \ell(I_k) \quad \forall k \in \mathbb{N}$$

Prop 6, monotonicity $\Rightarrow \forall n \in \mathbb{N}, \ \sum_{k=1}^{n} \ell(I_k) = m^*\left(\bigcup_{k=1}^{n} I_k\right) \leq m^*(\mathcal{U}) < \infty$.

independent of n

$\therefore \sum_{k=1}^{\infty} \ell(I_k) < \infty$.

Choose $n \in \mathbb{N} \ni \sum_{k=n+1}^{\infty} \ell(I_k) < \frac{\varepsilon}{2}$. $\mathcal{O} := \bigcup_{k=1}^{n} I_k$. $\because \mathcal{O} - E \subseteq \mathcal{U} \sim E$, monotonicity and $\Rightarrow$

$$m^*(\mathcal{O} \sim E) \leq m^*(\mathcal{U} \sim E) < \frac{\varepsilon}{2}.$$

$\because E \subseteq \mathcal{U}, \ E \sim \mathcal{O} \subseteq \mathcal{U} \sim \mathcal{O} = \bigcup_{k=n+1}^{\infty} I_k \Rightarrow m^*(E \sim \mathcal{O}) \leq \sum_{k=n+1}^{\infty} \ell(I_k) < \frac{\varepsilon}{2}$.

Thus, $m^*(\mathcal{O} \sim E) + m^*(E \sim \mathcal{O}) < \varepsilon$. $\square$

<u>Rmk.</u> Thm 11 (i) $\not\Rightarrow m^*(\mathcal{O} \sim E) < \varepsilon$ since $E \notin \mathcal{M} \Rightarrow m^*(\mathcal{O} \sim E) \neq m^*(\mathcal{O}) - m^*(E)$.

Defn. $m^*|_{\mathcal{M}}$ Lebesgue measure ; $E \in \mathcal{M} \Rightarrow m(E) = m^*(E)$.
$\overset{\parallel}{m}$

Prop 13. (Countable additivity) $\{E_k\}_{k=1}^{\infty}$; $E_k \in \mathcal{M}$ $\forall$ $k \in \mathbb{N}$, $E_i \cap E_j = \emptyset$ if $i \neq j$

Then $\bigcup_{k=1}^{\infty} E_k \in \mathcal{M}$ and $m\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} m(E_k)$

Pf: Prop 7 $\Rightarrow$ $\bigcup_{k=1}^{\infty} E_k \in \mathcal{M}$.

Prop 3 $\Rightarrow$ $m\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} m(E_k)$.

J.S.T. : $\sum_{k=1}^{\infty} m(E_k) \leq m\left(\bigcup_{k=1}^{\infty} E_k\right)$

Prop 6 $\Rightarrow$ $\forall$ $n \in \mathbb{N}$, $m\left(\bigcup_{k=1}^{n} E_k\right) = \sum_{k=1}^{n} m(E_k)$.

$\bigcup_{k=1}^{n} E_k \subseteq \bigcup_{k=1}^{\infty} E_k$ $\therefore$ Monotonicity $\Rightarrow$ $m\left(\bigcup_{k=1}^{\infty} E_k\right) \geq \sum_{k=1}^{n} m(E_k)$ $\forall$ $n \in \mathbb{N}$.

$$\underbrace{m\left(\bigcup_{k=1}^{\infty} E_k\right)}_{\text{independent of } n} \Rightarrow m\left(\bigcup_{k=1}^{\infty} E_k\right) \geq \sum_{k=1}^{\infty} m(E_k). \quad \square$$

Thm 14. The set fn Lebesgue measure, defined on the σ-algebra of Lebesgue measurable sets, assigns length to any interval, is translation invariant and is countably additive.

- $\{E_k\}_{k=1}^{\infty}$ → ascending if $E_k \subseteq E_{k+1}$ $\forall$ $k \in \mathbb{N}$
  → descending if $E_{k+1} \subseteq E_k$ $\forall$ $k \in \mathbb{N}$.

Thm 15 (Continuity of measure). (i) $\{A_k\}_{k=1}^{\infty} \subseteq \mathcal{M}$, $A_k \subseteq A_{k+1}$ $\forall$ $k \in \mathbb{N}$ then

$m\left(\bigcup_{k=1}^{\infty} A_k\right) = \lim_{k \to \infty} m(A_k)$.

(ii) $\{B_k\}_{k=1}^{\infty} \subseteq \mathcal{M}$, $B_{k+1} \subseteq B_k$ $\forall$ $k \in \mathbb{N}$ then $m\left(\bigcap_{k=1}^{\infty} B_k\right) = \lim_{k \to \infty} m(B_k)$.
$(m(B_1) < \infty)$

Pf. (i) <u>Case 1</u> $\exists\, k_0 \in \mathbb{N} \ni m(A_{k_0}) = \infty$; by monotonicity $m\left(\bigcup_{k=1}^{\infty} A_k\right) = \infty$ and $m(A_k) = \infty\ \forall$ $k \geq k_0$

<u>Case 2</u> $(m(A_k) < \infty\ \forall k)$ $A_0 = \phi$, $C_k = A_k \sim A_{k-1}\ \forall k \geq 1$ $\{A_k\}_{k=1}^{\infty} \uparrow$. $\therefore C_i \neq C_j$ if $i \neq j$ and $\bigcup_{k=1}^{\infty} A_k = \bigcup_{k=1}^{\infty} C_k$. By countable additivity of m,

$$m\left(\bigcup_{k=1}^{\infty} A_k\right) = m\left(\bigcup_{k=1}^{\infty} C_k\right) = \sum_{k=1}^{\infty} m(A_k \sim A_{k-1}).$$

$\because \{A_k\}_{k=1}^{\infty} \uparrow$ $\therefore$ $\sum_{k=1}^{\infty} m(A_k \sim A_{k-1}) = \sum_{k=1}^{\infty}\left[m(A_k) - m(A_{k-1})\right] = \lim_{n\to\infty} \sum_{k=1}^{n}\left[m(A_k) - m(A_{k-1})\right]$

$= \lim_{n\to\infty}\left[m(A_n) - m(A_0)\right]$.

$m(A_0) = m(\phi) = 0$.

(ii) $D_k = B_1 \sim B_k\ \forall k \in \mathbb{N}$. $\{B_k\}_{k=1}^{\infty} \downarrow \Rightarrow \{D_k\}_{k=1}^{\infty} \uparrow$.

(i) $\Rightarrow$ $m\left(\bigcup_{k=1}^{\infty} D_k\right) = \lim_{k\to\infty} m(D_k)$.

De Morgan $\Rightarrow$ $\bigcup_{k=1}^{\infty} D_k = \bigcup_{k=1}^{\infty}\left[B_1 \sim B_k\right] = B_1 \sim \bigcap_{k=1}^{\infty} B_k$.

Excision $\Rightarrow$ $m(D_k) = m(B_1) - m(B_k)$ $[\because m(B_k) < \infty]$ $\forall k \in \mathbb{N}$.

$\therefore$ $m\left(B_1 \sim \bigcap_{k=1}^{\infty} B_k\right) = \lim_{n\to\infty}\left[m(B_1) - m(B_n)\right]$.

$\|$

$m(B_1) - m\left(\bigcap_{k=1}^{\infty} B_k\right)$ by excision □

<u>Defn.</u> $E \in \mathcal{M}$; property holds <u>almost everywhere on</u> E (or holds for almost all $x \in E$) if $\exists\, E_0 \subseteq E \ni m(E_0) = 0$ & property holds $\forall\, x \in E \sim E_0$.

<u>Borel-Cantelli Lemma</u> $\{E_k\}_{k=1}^{\infty} \subseteq \mathcal{M} \ni \sum_{k=1}^{\infty} m(E_k) < \infty$. Then almost all $x \in \mathbb{R}$ belong to

at most finitely many of the E_k's.

__Pf.__ $\forall\, n \in \mathbb{N}$, $\quad m\left(\bigcup_{k=n}^{\infty} E_k\right) \leq \sum_{k=n}^{\infty} m(E_k) < \infty$. (by countable subadditivity)

$$m\left(\bigcap_{n=1}^{\infty}\left[\bigcup_{k=n}^{\infty} E_k\right]\right) = \lim_{n\to\infty} m\left(\bigcup_{k=n}^{\infty} E_k\right) \leq \lim_{n\to\infty} \sum_{k=n}^{\infty} m(E_k) = 0 \quad \text{(by continuity)}$$

$\therefore\ x \notin \bigcap_{n=1}^{\infty}\left[\bigcup_{k=n}^{\infty} E_k\right]\ \tilde{\forall}\ x \in \mathbb{R}$

$\therefore\ \tilde{\forall}\ x \in \mathbb{R}\ \exists\ n_x \in \mathbb{N} \ni x \in E_{i_1}, E_{i_2}, \ldots, E_{i_{n_x}}$ and $x \notin \{E_k\}_{k=1}^{\infty} \setminus \{E_{i_k} \mid k \in \{1, \ldots, n_x\}\}$. $\square$

__Properties:__

1. Finite additivity: $\{E_k\}_{k=1}^{n} \subseteq \mathcal{M} \ni E_i \cap E_j = \phi$ if $i \neq j$. Then $m\left(\bigcup_{k=1}^{n} E_k\right) = \sum_{k=1}^{n} m(E_k)$.

1'. Countable additivity (not inherited)

2. $A, B \in \mathcal{M}$; $A \subseteq B \Rightarrow m(A) \leq m(B)$ (montonicity)

3. (Excision) $A \subseteq B$, $m(A) < \infty$ then $m(B \sim A) = m(B) - m(A)$

$m(A) = 0 \Rightarrow m(B \sim A) = m(B)$

4. (Countable monotonicity) $\{E_k\}_{k=1}^{\infty} \subseteq \mathcal{M} \ni E \in \mathcal{M}$ and $E \subseteq \bigcup_{k=1}^{\infty} E_k$, then

$$m(E) \leq \sum_{k=1}^{\infty} m(E_k).$$

$E \subseteq \mathbb{R}$; $e_1, e_2 \in E$ rationally equivalent if $e_1 - e_2 \in \mathbb{Q}$ (equivalence relation)

$\hookrightarrow$ equivalence classes

- $\mathcal{F}$, non-empty family of non-empty sets.

Choice function $f : \mathcal{F} \to \bigcup_{F \in \mathcal{F}} F \ni \forall\ F \in \mathcal{F}$, $f(F) \in F$.

<u>Zermelo's axiom of choice.</u> $\mathcal{F}$, nonempty collection of nonempty sets. Then $\exists$ choice function on $\mathcal{F}$.

↳ Choice set $\mathcal{C}_E$ consisting of exactly one member of each equivalence class.

<u>Properties</u>:

(i) $c_1, c_2 \in \mathcal{C}_E \Rightarrow c_1 - c_2 \notin \mathbb{Q}$

(ii) $\forall\, x \in E\ \exists\ c \in \mathcal{C}_E \ni x = c + q$ where $q \in \mathbb{Q}$.

(i') $\forall\, \Lambda \subseteq \mathbb{Q}$, $\{\lambda + \mathcal{C}_E\}_{\lambda \in \Lambda}$ disjoint (or else $\lambda_1 - \lambda_2 \in \mathcal{C}_E$)

<u>Thm 17 (Vitali).</u> $E \subseteq \mathbb{R}$, $m^*(E) > 0$
↓
contains a subset that fails to be measurable.

<u>Lem 16.</u> $E^{\text{bdd}} \subseteq \mathbb{R}$, $E \in \mathcal{M}$. Suppose $\exists\ \Lambda^{\text{bdd, countably infinite}} \subseteq \mathbb{R} \ni \{\lambda + E\}_{\lambda \in \Lambda}$ disjoint. Then $m(E) = 0$.

<u>Pf lem.</u> $E \in \mathcal{M} \Rightarrow \lambda + E \in \mathcal{M}\ \forall\, \lambda \in \Lambda$. $\therefore$ $m\left[\bigcup_{\lambda \in \Lambda} (\lambda + E)\right] = \sum_{\lambda \in \Lambda} m(\lambda + E)$.

E, Λ bounded $\Rightarrow \bigcup_{\lambda \in \Lambda} (\lambda + E)$ bdd and $m\left[\bigcup_{\lambda \in \Lambda} (\lambda + E)\right] < \infty$. $\therefore$ LHS $< \infty$.

$m(\lambda + E) = m(E) \geq 0\ \forall\, \lambda \in \Lambda^{\text{countably infinite}}$. $\therefore \sum_{\lambda \in \Lambda} m(\lambda + E) < \infty \Rightarrow m(E) = 0$. □

<u>Pf thm.</u> By countable subadditivity of Outer Measure, suppose E bdd.

$\mathcal{C}_E$: any choice set for rational equivalence relation on E.

<u>Claim</u>: $\mathcal{C}_E \notin \mathcal{M}$.

Assume $\mathcal{C}_E \in \mathcal{M}$. $\Lambda_0^{\text{bdd, ctbly infinite}} \subseteq \mathbb{Q}$. $\{\lambda + \mathcal{C}_E\}_{\lambda \in \Lambda_0}$ disjoint $[(i')]$.

Lem $\Rightarrow m(\mathcal{C}_E) = 0$.

↓

$$m\left[\bigcup_{\lambda \in \Lambda_0} (\lambda + \mathcal{C}_E)\right] = \sum_{\lambda \in \Lambda_0} m(\lambda + \mathcal{C}_E) = 0.$$

$E \subseteq [-b,b]$. $\Lambda_0 := [-2b, 2b] \cap \mathbb{Q}$.

Claim: $E \subseteq \bigcup_{\lambda \in [-2b,2b]\cap\mathbb{Q}} (\lambda + \mathcal{C}_E)$

But $m(E) > 0$ while $m\left(\bigcup_{\lambda\in[-2b,2b]\cap\mathbb{Q}} (\lambda + \mathcal{C}_E)\right) = 0$.

$\rightarrow\leftarrow$

(ii) $x \in E$, $\exists\, c \in \mathcal{C}_E \ni x = c + q$, $q \in \mathbb{Q}$.

$x, c \in [-b, b] \Rightarrow q \in [-2b, 2b]$ □

Thm 18. $\exists\, A, B \subseteq \mathbb{R} \ni A \cap B = \emptyset$ & $m^*(A \cup B) < m^*(A) + m^*(B)$.

Proof. Else, every set is measurable $M = 2^{\mathbb{R}}$. □

☆ Cantor Set

$I = [0,1]$

$I \setminus \left(\frac{1}{3}, \frac{2}{3}\right) = \mathcal{C}_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$

$\mathcal{C}_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right] \ldots$ $\{\mathcal{C}_k\}_{k=1}^{\infty}$

$= \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{3}{9}\right] \cup \left[\frac{6}{9}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$

Cantor set, $\mathcal{C} = \bigcap_{k=1}^{\infty} \mathcal{C}_k$

$\{\mathcal{C}_k\}_{k=1}^{\infty}$: (i) descending sequence of closed sets;

(ii) $\forall\, k \in \mathbb{N}$, $\mathcal{C}_k$ disjoint union of 2^k closed intervals, each of length $\frac{1}{3^k}$

$\cancel{\mathcal{C}_k = \left[0, \frac{1}{3^k}\right] \cup \left[\frac{2}{3^k}, \frac{1}{3^{k-1}}\right] \cup \left[\frac{2}{3^{k-1}}, \frac{7}{3^k}\right] \cup}$

$$\mathcal{C} = [0,1] \setminus \bigcup_{n=0}^{\infty} \bigcup_{k=0}^{3^n - 1} \left(\frac{3k+1}{3^{n+1}}, \frac{3k+2}{3^{n+1}}\right) = \bigcap_{n=1}^{\infty} \bigcup_{k=0}^{3^{n-1}-1} \left(\left[\frac{3k+0}{3^n}, \frac{3k+1}{3^n}\right] \cup \left[\frac{3k+2}{3^n}, \frac{3k+3}{3^n}\right]\right)$$

Prop 19. $\mathcal{C}$ closed, uncountable; $m(\mathcal{C}) = 0$.

Pf. $\mathcal{C} = \bigcap_{k=1}^{\infty} C_k^{\text{closed}} \Rightarrow \mathcal{C}$, closed.

$\therefore \mathcal{C} \in \mathcal{M}$. $(C_k \in \mathcal{M} \ \forall k \in \mathbb{N})$.

C_k =, disjoint union of 2^k intervals, each of length $\frac{1}{3^k}$. By finite additivity,

$$m(C_k) = \left(\frac{2}{3}\right)^k.$$

Monotonicity $\Rightarrow$ $m(\mathcal{C}) \le m(C_k) = \left(\frac{2}{3}\right)^k \ \forall k \in \mathbb{N} \Rightarrow m(\mathcal{C}) = 0$.

Let $\mathcal{C}$ countable.

enumeration $\{c_k\}_{k=1}^{\infty}$.

$c_1 \in C_1 = F_1 \cup F_1'$. Let $c_1 \in F_1'$. $C_2 = \underbrace{F_2 \cup F_2'}_{F_1} \cup F_2'' \cup F_2'''$; $c_2 \in F_2' \cup F_2'' \cup F_2'''$

~~$C_2 \in C_2 = F_2 \cup F_2' \cup F_2'' \cup F_2'''$. Let $c_2 \in F_2' \cup F_2'' \cup F_2'''$~~

... $\hookrightarrow \{F_k\}_{k=1}^{\infty}$ $\forall k \in \mathbb{N}$

(i) F_k closed and $F_{k+1} \subseteq F_k$

(ii) $F_k \subseteq C_k$

(iii) $c_k \notin F_k$

(i), Nested Set Thm $\Rightarrow \bigcap_{k=1}^{\infty} F_k \neq \phi$

$$x \in \bigcap_{k=1}^{\infty} F_k \subseteq \bigcap_{k=1}^{\infty} C_k = \mathcal{C}$$

$\therefore x = c_n$ for some $n \in \mathbb{N}$

$c_n = x \in \bigcap_{k=1}^{\infty} F_k \subseteq F_n$. $\rightarrow\leftarrow$

Defn. $f : \underset{\subseteq \mathbb{R}}{U} \to \mathbb{R}$

$f \uparrow$ if $u \le v \Rightarrow f(u) \le f(v)$

$f \Uparrow$ if $u < v \Rightarrow f(u) < f(v)$.

strictly increasing

Cantor-Lebesgue fn

cts ; $\uparrow$, $\varphi(1) > \varphi(0)$ but derivative exists and is 0 on a set of measure 1.

$k \in \mathbb{N}$; O_k : union of the $2^k - 1$ intervals which have been removed in the first k stages of Cantor deletion process.

$\mathbb{C}_k = [0,1] \sim \mathcal{O}_k$

$\mathcal{O} = \bigcup_{k=1}^{\infty} \mathcal{O}_k$; De Morgan $\Rightarrow \mathbb{C} = [0,1] \sim \mathcal{O}$.

<u>$k \in \mathbb{N}$</u> $\quad \varphi\big|_{\mathcal{O}_k}$ $\quad 2^k - 1$ open intervals

values : $\left\{ \frac{1}{2^k}, \frac{2}{2^k}, \dots, \frac{2^k-1}{2^k} \right\}$

<u>Step 1</u>. $\varphi(x) = \frac{1}{2}$ if $x \in \left(\frac{1}{3}, \frac{2}{3}\right) = \mathcal{O}_1$

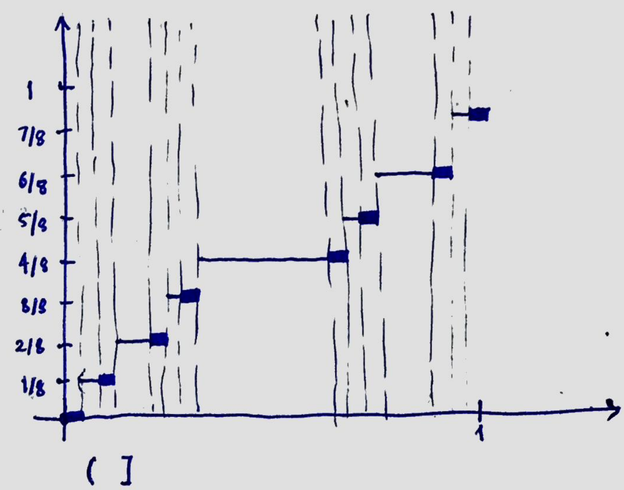
<u>Step 2</u>

$$\left.\varphi(x) = \begin{cases} \frac{1}{4} & \text{if } x \in \left(\frac{1}{9}, \frac{2}{9}\right) \\ \frac{2}{4} & \text{if } x \in \left(\frac{3}{9}, \frac{6}{9}\right) = \mathcal{O}_1 \\ 3/4 & \text{if } x \in \left(\frac{7}{9}, \frac{8}{9}\right) \end{cases}\right] \mathcal{O}_2 .$$

<u>Step 3</u>.

$$\varphi(x) = \begin{cases} \frac{1}{8} & \text{if } x \in \left(\frac{1}{27}, \frac{2}{27}\right) \\ \frac{2}{8} & \text{if } x \in \left(\frac{3}{27}, \frac{6}{27}\right) \\ \frac{3}{8} & \text{if } x \in \left(\frac{7}{27}, \frac{8}{27}\right) \\ \frac{4}{8} & \text{if } x \in \left(\frac{9}{27}, \frac{18}{27}\right) = \mathcal{O}_1 \\ \frac{5}{8} & \text{if } x \in \left(\frac{19}{27}, \frac{20}{27}\right) \\ \frac{6}{8} & \text{if } x \in \left(\frac{21}{27}, \frac{24}{27}\right) \\ \frac{7}{8} & \text{if } x \in \left(\frac{25}{27}, \frac{26}{27}\right) \end{cases}$$

$\mathcal{O}_2$ $\quad \mathcal{O}_3$



$\varphi : \mathbb{C} \longrightarrow [0,1]$

$\varphi(0) = 0$

$\varphi(x) = \sup\{\varphi(t) \mid t \in \mathcal{O} \cap [0,x)\}$

if $x \in \mathbb{C} \sim \{0\}$.

Prop 20. $\varphi : [0,1] \xrightarrow{\text{onto}} [0,1]$ $\uparrow$, cts | $\exists\ \varphi'$ on open set $\mathcal{O}_0 = [0,1] \sim \mathcal{C}$ | $\varphi' = 0$ on $\mathcal{O}$, $m^*(\mathcal{O}) = 1$

Pf. $\boxed{\uparrow}$ $\varphi|_{\mathcal{O}} \uparrow \Rightarrow \varphi|_{[0,1]} \uparrow$

$\boxed{\text{cty}}$ cts at each pt in $\mathcal{O}$ since each such pt belongs to the open interval on which it is constant.

$x_0 \in \mathcal{C}$; $x_0 \notin \{0,1\}$.

$x_0 \in \mathcal{C} \Rightarrow x_0 \notin \mathcal{O} \Rightarrow x_0 \notin \mathcal{O}_k$ for sufficiently large k.

Let $a_k, b_k \in$ consecutive intervals in $\mathcal{O}_k \ni a_k < x_0 < b_k$

$\varphi(b_k) - \varphi(a_k) = \frac{1}{2^k}$

k arbitrarily large $\Rightarrow \varphi$ fails to have a jump discontinuity at x_0.

: for increasing function, the only possible type of discnty.

$\therefore \varphi$ cts at x_0.

$x_0 \in \{0,1\}$: similar argument.

$\boxed{\varphi' = 0}$ Since φ is constant on each of the intervals removed at any stage of the removal process, its derivative exists and equals 0 at each pt in $\mathcal{O}$.

$\boxed{m(\mathcal{O}) = 1}$ $m(\mathcal{C}) = 0 \Rightarrow m(\mathcal{O}) = m([0,1]) - m(\mathcal{C}) = 1$

$\varphi(0) = 0$, $\varphi(1) = 1$, $\varphi^{\text{cts}} \uparrow$; by Intermediate value thm, φ maps $[0,1] \xrightarrow{\text{onto}} [0,1]$.

Prop 21. φ, Cantor Lebesgue function

$\psi : [0,1] \to \mathbb{R}$, $\psi(x) = \varphi(x) + x \ \forall x \in [0,1]$. Then, $\psi \uparrow_{\text{strictly}}$, ψ cts and $\psi : [0,1] \xrightarrow{\text{onto}} [0,2]$

(i) maps the Cantor set $\mathbf{C}$ onto a measurable set of positive measure;

(ii) maps a measurable set, a subset of the Cantor set, onto a nonmeasurable set.

Pf. (i)

- $\psi(x) = \underset{\text{cts}}{\underset{\uparrow}{\varphi(x)}} + \underset{\text{cts}}{\underset{\uparrow}{x}} \Rightarrow \psi(x)$ cts

- $\varphi(x)\uparrow$, $x \Uparrow \Rightarrow \psi(x) \Uparrow$.

- $\psi(0) = 0$, $\psi(1) = 2$, $\psi([0,1]) = [0,2]$.

  $\mathcal{O} = [0,1] \sim \mathbf{C} \Rightarrow [0,1] = \mathbf{C} \cup \mathcal{O}$ disjoint decomposition

  $\xrightarrow{\text{lifting}}$ $[0,2] = \psi(\mathbf{C}) \cup \psi(\mathcal{O})$

A strictly increasing cts function defined on an interval has a cts inverse.

$\underset{\mathbf{C}^{\text{closed}}}{\psi(\mathbf{C})^{\text{closed}}}, \underset{\mathcal{O}^{\text{open}}}{\psi(\mathcal{O})^{\text{open}}} \Rightarrow \psi(\mathbf{C}), \psi(\mathcal{O}) \in \mathcal{M}$

<u>T.S.T.:</u> $m(\psi(\mathcal{O})) = 1$.

$\{I_k\}_{k=1}^{\infty}$ enumeration of the collection of intervals removed in the Cantor removal process.

$\mathcal{O} = \bigcup_{k=1}^{\infty} I_k$.

$\varphi|_{I_k}$ is constant $\forall k \in \mathbb{N} \Rightarrow \psi$ maps I_k onto a translated copy of itself of the same length. ψ one-to-one $\Rightarrow$ ~~$\{I_k\}_{k=1}^{\infty}$~~ $\{\psi(I_k)\}_{k=1}^{\infty}$ disjoint. By countable additivity of measure,

$$m(\psi(\mathcal{O})) = \sum_{k=1}^{\infty} \ell(\psi(I_k)) = \sum_{k=1}^{\infty} \ell(I_k) = m(\mathcal{O}).$$

$\therefore m(\mathbf{C}) = 0 \Rightarrow m(\mathcal{O}) = 1$. $\therefore m(\psi(\mathcal{O})) = 1 \Rightarrow m(\psi(\mathbf{C})) = 2 - m(\psi(\mathcal{O})) = 1$.

(ii) Vitali's Thm $\Rightarrow \exists\, W \subseteq \psi(\mathbf{C}) \ni W \notin \mathcal{M}$.

$\psi^{-1}(W) \in \mathcal{M}$ and $m(\psi^{-1}(W)) = 0$ since $\psi^{-1}(W) \subseteq \mathbf{C}$.

$\psi^{-1}(W)$ is a measurable subset of the Cantor set, which is mapped by ψ onto a nonmeasurable set. $\square$

Prop 22. $\exists$ a measurable set, a subset of the Cantor set that is not a Borel set.

Pf: $\exists$ $A \subseteq [0,1]$ $\ni$ $A \in \mathcal{M}$ and $\psi(A) = W \notin \mathcal{M}$.

Prob 47 $\Rightarrow$ a strictly increasing cts function defined on an interval maps Borel sets onto Borel sets.

$\therefore$ $A \notin \mathcal{B}$ $[A \in \mathcal{B} \Rightarrow \psi(A) \in \mathcal{B} \Rightarrow \psi(A) \in \mathcal{M} \rightarrow\leftarrow]$. $\square$

Generalized Cantor set

$F \subseteq [0,1]$

each intervals deleted at the nth deletion stage was length $\alpha 3^{-n}$ $(0<\alpha<1)$.

1. F closed

2. $[0,1] \sim F$ dense in $[0,1]$

3. $m(F) = 1-\alpha$

4. Complement's boundary (open set of real nos.) has positive measure.

Proof: Let F_k denote the set of points that remain after k removal operations. F_k is the union of 2^k disjoint closed intervals, each of length $\ell_k := 2^{-k}\left(1-\alpha+\alpha\left(\frac{2}{3}\right)^k\right)$. Since a finite union of closed sets is closed, each F_k is closed. Since an intersection of closed sets is closed and $F = \bigcap_{k=1}^{\infty} F_k$, F is closed. Define $O = [0,1] \sim F$ and pick $x, y \in [0,1]$. Since O is open, if either x or y belongs to O we can find a point between x and y which also belongs to O. So suppose both x and y belong to F. Choose $k \in \mathbb{N} \ni \ell_k < |x-y|$. Write $F_k = \bigcup_{n=1}^{2^k} I_n$, where each I_n is a closed interval of length ℓ_k. Since x and y both belong to F_k, they must belong to one of the intervals in $\{I_n\}_{n=1}^{2^k}$. However x and y cannot belong to the same interval since $\ell_k < |x-y|$. Since the intervals are disjoint, there must exist a point between x and y which is not in F_k, therefore not in F. Thus O is dense in $[0,1]$. Finally, observe that O is countable union of the disjoint collection of open intervals which are removed during the construction of F. At the k^{th} deletion stage, 2^{k-1} intervals of length $\alpha 3^{-k}$ are removed.

$$\therefore \quad m(O) = \frac{\alpha}{2} \sum_{k=1}^{\infty} \left(\frac{2}{3}\right)^k = \alpha,$$

which implies $m(F) = 1-\alpha$ by the excision property.

Since O is open, it does not contain any of its boundary points. But since O is dense in $[0,1]$, every point in F is a boundary point of O. Thus $\partial O = F$ and has measure $1-\alpha$. $\square$

① m set fn defined $\forall$ sets in a σ-algebra $\mathcal{A}$, with values in $[0,\infty]$
↓
countably additive over countable disjoint collections of sets in $\mathcal{A}$.

1 $A, B \in \mathcal{A} \ni A \subseteq B$; then $m(A) \leq m(B)$. monotonicity.

Soln $m(B) = m((B \setminus A) \cup A) = m(B \setminus A) + m(A) \geq m(A)$.

2 P.t., if $\exists A \in \mathcal{A} \ni m(A) < \infty$, then $m(\phi) = 0$.

Soln. $m(A) = m(A \cup \phi) = m(A) + m(\phi) \Rightarrow m(\phi) = 0$ if $m(A) < \infty$.

3. $\{E_k\}_{k=1}^{\infty} \subseteq \mathcal{A}$. P.t. $m\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} m(E_k)$.

Soln. $E_1' = E_1$; $E_{n+1}' = E_{n+1} \sim \bigcup_{i=1}^{n} E_i$ $(n \geq 1)$

↳ $E_n' \subseteq E_n$ | $\bigcup_{n=1}^{\infty} E_n' = \bigcup_{n=1}^{\infty} E_n$

$\{E_n'\}_{n=1}^{\infty}$ pairwise disjoint

$$\Rightarrow m\left(\bigcup_{n=1}^{\infty} E_n\right) = m\left(\bigcup_{n=1}^{\infty} E_n'\right) = \sum_{n=1}^{\infty} m(E_n') \leq \sum_{n=1}^{\infty} m(E_n).$$

4. c, set fn defined on all subsets of $\mathbb{R}$ | s.t. c countably additive, translation invariant

$$c(E) = \begin{cases} \infty, & |E| = \infty \\ \cancel{c(E)}\ |E|, & |E| < \infty \\ 0, & E = \phi \end{cases}$$

Counting measure

Soln. Since if $E_i \cap E_j = \phi \ \forall i \neq j$, $\left|\bigcup_{i=1}^{\infty} E_i\right| = \sum_{i=1}^{\infty} |E_i|$.

5. Using properties of outer measure, p.t. the interval $[0,1]$ is not countable

Soln. Any countable set has outer measure 0. $m^*([0,1]) = 1$

6. $A = [0,1] \cap (\mathbb{R} \setminus \mathbb{Q})$. P.t. $m^*(A) = 1$

Soln. $\mathbb{Q}$ countable $\Rightarrow \mathbb{Q}\cap[0,1] \subset \mathbb{Q}$ countable.

Again, $[0,1] = (\mathbb{Q}\cap[0,1]) \cup ((\mathbb{R}\setminus\mathbb{Q})\cap[0,1])$ and $(\mathbb{Q}\cap[0,1]) \cap ((\mathbb{R}\setminus\mathbb{Q})\cap[0,1]) = \phi$.

$\therefore$ By countable subadditivity, $1 = m^*([0,1]) \le m^*([0,1]\sim A) + m^*(A) = m^*(A)$ where $A = (\mathbb{R}\setminus\mathbb{Q})\cap[0,1]$. But $m^*(A) \le m^*([0,1]) \Rightarrow m^*(A) = 1$.

7. $A \subseteq \mathbb{R}$ $\downarrow$ G_δ set if $A = \bigcap_{i=1}^{\infty} U_i^{\text{open}}$ | S.t. $\forall\, E^{\text{bounded}}$, $\exists\, G_\delta$ set $G \ni E \subseteq G$ and $m^*(G) = m^*(E)$.

Soln. E, bounded $\Rightarrow m^*(E)$ finite $\Rightarrow \forall\, k\ \exists\ \{I_{k,n}^{\text{open, bdd}}\}_{n=1}^{\infty} \ni E \subseteq \bigcup_{n=1}^{\infty} I_{k,n}$ and $m^*(E) + \frac{1}{k} > \sum_{n=1}^{\infty} \ell(I_{k,n})$.

$$G_k = \bigcup_{n=1}^{\infty} I_{k,n} \quad ; \quad G := \bigcap_{n=1}^{\infty} G_k, \ G_\delta\text{-set}.$$

$\downarrow$ open $\forall\, k$

$m^*(G_k) \le \sum_{n=1}^{\infty} \ell(I_{k,n})$. Since $E \subseteq G \subseteq G_k$,

$$\underbrace{m^*(E) \le m^*(G) \le m^*(G_k) \le \sum_{n=1}^{\infty} \ell(I_{k,n}) < m^*(E) + \frac{1}{k}}_{\text{holds } \forall k \ \Rightarrow\ m^*(E) = m^*(G).}, \text{ by monotonicity of } m^*.$$

holds $\forall\, k \Rightarrow m^*(E) = m^*(G)$.

8. $B = \mathbb{Q}\cap[0,1]$.

$\{I_k^{\text{open}}\}_{k=1}^{n} \ni B \subseteq \bigcup_{k=1}^{n} I_k$. P.t. $\sum_{k=1}^{n} m^*(I_k) \ge 1$.

Soln. (i) $\underline{\bigcup_{k=1}^{n} I_k^{\text{open}} \supsetneq [0,1]}$. If $\bigcup_{k=1}^{n} I_k \subset [0,1]$ then $\inf\left(\bigcup_{k=1}^{n} I_k\right) = \alpha > 0$ or $\sup\left(\bigcup_{k=1}^{n} I_k\right) = \beta < 1$. WLOG in the former case, $\exists\, r \in \mathbb{Q} \ni 0 \le r < \alpha$ since $\mathbb{Q}$ is dense in $\mathbb{R}$. Also $\bigcup_{k=1}^{n} I_k \ne [0,1]$ since $\bigcup_{k=1}^{n} I_k$ is open.

Let $\exists\ D \subseteq [0,1] \ni m^*(D) > 0$ and
$\qquad\qquad \| \ \neq \phi$ (any)

• A_i $(i=1,\dots,n)$ finite colln of sets of real nos. $B = \bigcup_{i=1}^n A_i$; $\bar{B} = \bigcup_{i=1}^n \bar{A}_i$.

↳ $x \in \bigcup_{i=1}^n \bar{A}_i$; $x \in I$ open interval

$\Downarrow$

$x \in \bar{A}_i$ for some i $\Rightarrow \exists\ y \in I \ni y \in A_i$.

$\Rightarrow y \in B \Rightarrow x \in \bar{B}$ $\quad \therefore \bigcup_{i=1}^n \bar{A}_i \subseteq \bar{B}$.

<u>Reverse inclusion</u>

$x \in \bigcap_{i=1}^n \bar{A}_i^{\,c}$

$\therefore \forall i, \exists\ \varepsilon_i > 0 \ni (x-\varepsilon_i, x+\varepsilon_i) \subseteq A_i^c$

$\varepsilon = \min_i \varepsilon_i$. Then $(x-\varepsilon, x+\varepsilon) \subseteq A_i^c\ \forall i \Rightarrow (x-\varepsilon, x+\varepsilon) \subseteq \bigcap_{i=1}^n A_i^c = B^c$.

$\therefore x \in \bar{B}^c$. Thus, $\bigcap_{i=1}^n \bar{A}_i^{\,c} \subseteq B^c \Rightarrow \bar{B} \subseteq \bigcup_{i=1}^n \bar{A}_i$.

▷ $\underset{\text{dense}}{Q \subseteq R}$, $\bar{B} = [0,1]$. $\therefore [0,1] = \bar{B} \subseteq \overline{\bigcup_{k=1}^n I_k} = \bigcup_{k=1}^n \bar{I}_k$.

Prop 1, monotonicity and subadditivity of outer measure $\Rightarrow$

$$1 \leq m^*([0,1]) \leq m^*\left(\bigcup_{k=1}^n \bar{I}_k\right) \leq \sum_{k=1}^n m^*(\bar{I}_k) = \sum_{k=1}^n m^*(I_k).$$

9. <u>$m^*(A) = 0 \Rightarrow m^*(A \cup B) = m^*(B)$.</u>

$m^*(A \cup B) \leq m^*(A) + m^*(B) = m^*(B)$. Again, $m^*(B) \leq m^*(A \cup B)$ since $B \subseteq A \cup B$.

$\therefore m^*(A \cup B) = m^*(B)$.

10. $A^{bdd}, B^{bdd} \ni \exists\, \alpha > 0 \ni |a-b| \geq \alpha \;\forall a \in A, b \in B$. P.t. $m^*(A \cup B) = m^*(A) + m^*(B)$.

<u>Soln.</u> <u>Subadditivity</u> $\Rightarrow m^*(A \cup B) \leq m^*(A) + m^*(B)$.

fix $\varepsilon > 0$. A bdd, B bdd $\Rightarrow (A \cup B)^{bdd} \Rightarrow m^*(A \cup B) < \infty$.

$\therefore \exists \left\{ I_k^{\text{open, bdd}} \right\}_{k=1}^{\infty}$ $(\neq \phi)$ $\ni$ $A \cup B \subseteq \bigcup_{k=1}^{\infty} I_k$ and $m^*(A \cup B) > \sum_{k=1}^{\infty} \ell(I_k) - \varepsilon$.

WLOG, $|I_k| < \frac{\alpha}{2} \quad \forall k \in \mathbb{N}$

$\forall k \in \mathbb{N}$ either $A \cap I_k \neq \phi$, $B \cap I_k = \phi$ or $A \cap I_k = \phi$, $B \cap I_k \neq \phi$.

$\mathcal{A} = \{k : I_k \cap A \neq \phi\}$, $\quad \mathcal{B} = \{k : I_k \cap B \neq \phi\}$.

$\{I_k\}_{k \in \mathcal{A}}$ open cover of A ; $\{I_k\}_{k \in \mathcal{B}}$ open cover of B.

$$\underbrace{m^*(A \cup B) > \sum_{k \in \mathcal{A}} \ell(I_k) + \sum_{k \in \mathcal{B}} \ell(I_k) - \varepsilon \geq m^*(A) + m^*(B) - \varepsilon}$$

holds $\forall \varepsilon > 0 \Rightarrow m^*(A \cup B) \geq m^*(A) + m^*(B)$.

$\therefore m^*(A \cup B) = m^*(A) + m^*(B)$.

11. σ-algebra in $\mathbb{R}$ contains all intervals of form $(a, \infty) \Rightarrow$ contains all intervals.

<u>Soln.</u> $(-\infty, a) = (a, \infty)^c$

$(a, b) = (-\infty, b) \cap (a, \infty)$.

$[a, \infty) = \bigcap_{n=1}^{\infty} \left(a - \frac{1}{n}, \infty\right)$ etc.

12. S.t. every interval is a Borel set.

<u>Soln.</u> Open intervals are anyhow Borel sets.

$[a, b] = \bigcap_{n=1}^{\infty} \left(a - \frac{1}{n}, b + \frac{1}{n}\right) \quad (n \in \mathbb{N})$ | since $\mathcal{B}$ is a σ-algebra.

$[a, b) = \bigcap_{n=1}^{\infty} \left(a - \frac{1}{n}, b\right)$ etc.

13. S.t (i) translate of an F_σ set is also F_σ.

(ii) translate of a G_δ set also G_δ.

(iii) translate of a measure 0 set also has measure 0.

<u>Soln</u> (i) $F = \bigcup_{n=1}^{\infty} F_n^{\text{closed}} \Rightarrow F+y = \bigcup_{n=1}^{\infty} (F_n+y)^{\text{closed}}$.

$(y \in \mathbb{R})$

(ii) $G = \bigcap_{n=1}^{\infty} G_n^{\text{open}} \Rightarrow G+y = \bigcap_{n=1}^{\infty} (G_n+y)^{\text{open}}$ $[y \in \mathbb{R}]$.

(iii) since outer measure is translation invariant.

14. S.t. if $m^*(E) > 0$ then $\exists$ bounded subset of E that also has positive outer measure.

<u>Soln.</u> Suppose every bounded subset of E has outer measure 0. Then $\forall\, n$,

$$\sum_{i=1}^{n} m^*((-n,n) \cap E) = 0 \Rightarrow \sum_{n=1}^{\infty} m^*((-n,n) \cap E) = 0.$$

By the countable subadditivity of m^*,

$$m^*(E) = m^*\left(\bigcup_{n=1}^{\infty} (-n,n) \cap E\right) \leq \sum_{n=1}^{\infty} m^*((-n,n) \cap E) = 0. \quad \rightarrow\leftarrow$$

15. Show that if E has finite measure and $\varepsilon > 0$,
then E is a disjoint union of finite no. of measurable sets, each of which has measure $\leq \varepsilon$.

<u>Soln</u> Let k_n denote an enumeration of integers. $I_n := [k_n \cdot \varepsilon, (k_n+1) \cdot \varepsilon)$.

$\{I_n\}_{n=1}^{\infty}$ disjoint and measurable $\subseteq \mathcal{M}$, $I_i \cap I_j = \phi$ if $j \neq i$.

$$m^*\left(\bigcup_{n=1}^{N}(E_n \cap I_n)\right) = \sum_{n=1}^{N} m^*(E \cap I_n) \quad \forall\, N \in \mathbb{N}.$$

$\because m^*\left(\bigcup_{n=1}^{N}(E \cap I_n)\right) \leq m^*(E) < \infty$ by monotonicity, converges.

$\therefore \exists\ N \ni \sum_{n=N+1}^{\infty} m^*(E \cap I_n) < \varepsilon. \Rightarrow m^*\left(E \cap \bigcup_{n=N+1}^{\infty} I_n\right) < \varepsilon$ by subadditivity

$E_0 = E \cap \bigcup_{n=N+1}^{\infty} I_n$, $E_n = E \cap I_n \quad \forall\ n \in \{1, \ldots, N\}$.

$\therefore E_i \cap E_j = \phi$ if $i \neq j$ $(0 \leq i, j \leq N)$, $\{E_n\}_{n=0}^{N} \subseteq \mathcal{M}$ and $m^*(E_n) \leq \varepsilon\ \forall\ n \in \{0, 1, \ldots, N\}$.

16. Thm 1. $(0) \Leftrightarrow (iii) \Leftrightarrow (iv)$

Soln. $(0) \Rightarrow (iii)$
$E \in \mathcal{M} \Rightarrow \forall\ \varepsilon > 0,\ \exists\ \mathcal{O}^{\text{open}} \ni E^c \subseteq \mathcal{O}\ \&\ m^*(\mathcal{O} \sim E^c) < \varepsilon$.

$\therefore E^c \in \mathcal{M}$, $(\mathcal{O}^c)^{\text{closed}}$; $\mathcal{O}^c \subseteq E$. ~~and m^*~~

$m^*(\mathcal{O} \sim E^c) < \varepsilon \quad$ T.S.T. $m^*(E \sim \mathcal{O}^c) < \varepsilon$

Now, $m^*(\mathcal{O} \sim E^c) < \varepsilon \Rightarrow m^*(E \sim \mathcal{O}^c) < \varepsilon$.

$\mathcal{O} \sim E^c = \mathcal{O} \cap E = E \cap \mathcal{O} = E \sim \mathcal{O}^c$

$\mathcal{O} \sim E^c = \mathcal{O} \cap E$
$\Rightarrow m^*(\mathcal{O} \cap E) < \varepsilon$
$\quad \| $
$\Rightarrow m^*(E \cap \mathcal{O}) < \varepsilon$
$\quad \|$
$m^*(E \sim \mathcal{O}^c)$

$(iii) \Rightarrow (iv)$. $\forall\ k \in \mathbb{N},\ \exists\ F_k \subseteq E,\ m^*(E \sim F_k) < \frac{1}{k}$

$F = \bigcup_{k=1}^{\infty} F_k$, F_σ-set; $F \subseteq E$.

$m^*(E \sim F) \leq m^*(E \sim F_k) < \frac{1}{k} \Rightarrow m^*(E \sim F) = 0$.

$(iv) \Rightarrow (0)$. $E = F \cup (E \sim F)$.

17. S.t. $E \in \mathcal{M}$ iff $\forall\ \varepsilon > 0, \exists\ F^{\text{closed}}, \mathcal{O}^{\text{open}} \ni F \subseteq E \subseteq \mathcal{O}\ \&\ m^*(\mathcal{O} \sim F) < \varepsilon$.

Soln. $E \in \mathcal{M} \Rightarrow \forall\ \varepsilon > 0, \exists\ F^{\text{closed}}, \mathcal{O}^{\text{open}} \ni F \subseteq E \subseteq \mathcal{O}\ \&\ m^*(\mathcal{O} \sim F) < \varepsilon$.

By monotonicity, $m^*(\mathcal{O} \sim F) < \varepsilon \Rightarrow m^*(\mathcal{O} \sim E) < \varepsilon$, $m^*(E \sim F) < \varepsilon$

$\forall\, \varepsilon>0\ \exists\ \mathcal{O}^{\text{open}} \ni m^*(\mathcal{O}\sim E)<\varepsilon$

$\forall\, k\in\mathbb{N}$, choose $\mathcal{O}_k^{\text{open}} \ni E\subseteq \mathcal{O}_k$ and $m^*(\mathcal{O}_k - E)<\frac{1}{k}$.

$G := \bigcap_{k=1}^{\infty} \mathcal{O}_k \supseteq E$ | $m^*(G\sim E)\le m^*(\mathcal{O}_k\sim E)<\frac{1}{k} \Rightarrow m^*(G\sim E)=0$

$\downarrow$

G_δ | $E = G\cap (G\sim E)^c \Rightarrow E\in\mathcal{M}$.

$\underset{G_\delta,\ \in\mathcal{M}}{\underset{\downarrow}{G}}\ \cap\ \underbrace{(G\sim E)^c}_{\in\mathcal{M}}$

17. $m^*(E)<\infty$.

S.t. $E\notin\mathcal{M} \Rightarrow \exists\ \mathcal{O}^{\text{open}} \ni E\subseteq\mathcal{O} \ni m^*(\mathcal{O})<\infty$ and $m^*(\mathcal{O}\sim E)>m^*(\mathcal{O})-m^*(E)$.

Soln. Thm 11 (i) $E\notin\mathcal{M} \Rightarrow \exists\ \varepsilon_0>0 \ni m^*(\mathcal{O}\sim E)\ge\varepsilon_0$ for $\mathcal{O}^{\text{open}}\ni E\subseteq\mathcal{O}$.

$\because m^*(E)<\infty$, $\exists\ \{I_k^{\text{open interval}}\}_{k=1}^{\infty} \ni E\subseteq\bigcup_{k=1}^{\infty} I_k$ and $m^*(E)>\sum_{k=1}^{\infty} \ell(I_k)-\varepsilon_0$

Let $\mathcal{O}=\bigcup_{k=1}^{\infty} I_k$. Then

$$m^*(\mathcal{O})-m^*(E)\le\sum_{k=1}^{\infty}\ell(I_k)-m^*(E)<\varepsilon_0\le m^*(\mathcal{O}\sim E).$$

18. $m^*(E)<\infty$. S.t. $\exists\ F_\sigma$-set F and G_δ-set $G \ni F\subseteq E\subseteq G$ and $m^*(F)=m^*(E)=m^*(G)$.

~~19. $m^*(E)<\infty$.~~ Soln. $\boxed{\text{Prob 7}}$ $m^*(E)<\infty$. Construct a G_δ-set $G \ni E\subseteq G$ and $m^*(E)=m^*(G)$.

$E\in\mathcal{M}$ iff $\exists\ F_\sigma$ set $F \ni F\subseteq E$ and $m^*(F)=m^*(E)$.

20. (Lebesgue) $m^*(E)<\infty$. S.t. $E\in\mathcal{M}$ iff $\forall$ open, bdd (a,b)

$$b-a = m^*((a,b)\cap E)+m^*((a,b)\sim E).$$

Soln. $E\in\mathcal{M}$. Then $b-a=m^*((a,b))=m^*((a,b)\cap E)+m^*((a,b)\sim E)$.

Converse. Fix $A \ni m^*(A)<\infty$. $\forall\ \varepsilon>0$, we can choose $\{\underset{\substack{\uparrow\\ \text{open, bdd}}}{(a_k,b_k)}\}_{k=1}^{\infty} \ni A\subseteq\bigcup_{k=1}^{\infty}(a_k,b_k)$

and $m^*(A)>\sum_{k=1}^{\infty}(b_k-a_k)-\varepsilon$. Then,

$$m^*(A) > \sum_{k=1}^{\infty} \left(m^*((a_k, b_k) \cap E) + m^*((a_k, b_k) \sim E)\right) - \varepsilon$$

$$\geq m^*\left(\bigcup_{k=1}^{\infty} (a_k, b_k) \cap E\right) + m^*\left(\bigcup_{k=1}^{\infty} (a_k, b_k) \sim E\right) - \varepsilon$$

$$\geq m^*(A \cap E) + m^*(A \sim E) - \varepsilon$$

holds $\forall\ \varepsilon > 0$. $\therefore$ $m^*(A) \geq m^*(A \cap E) + m^*(A \sim E)$.

21. Thm 11 (ii) primitive defn of measurable set

P.t. union of 2 mble sets is mble.

Soln Let $E_1, E_2 \in \mathcal{M}$ $\therefore \exists\ G_1, G_2\ (G_\delta) \ni$ ~~$m^*(G_1$~~ $E_i \subseteq G_i$ and $m^*(G_i \sim E_i) = 0$ $(i \in \{1,2\})$.

$$0 = m^*(G_1 \sim E_1) + m^*(G_2 \sim E_2) \geq m^*((G_1 \sim E_1) \cup (G_2 \sim E_2)) = m^*\left[((G_1 \cup G_2) \sim (E_1 \cup E_2)) \cup (G_1 \cap E_2) \cup (G_2 \cap E_1)\right]$$

$$\geq m^*((G_1 \cup G_2) \sim (E_1 \cup E_2)) \geq 0.$$

$\therefore$ $m^*((G_1 \cup G_2) \sim (E_1 \cup E_2)) = 0$.

Now, $G_1 = \bigcap_{i=1}^{\infty} U_i^{\text{open}}$, $G_2 = \bigcap_{i=1}^{\infty} V_i^{\text{open}} \Rightarrow G_1 \cup G_2 = \left(\bigcap_{i=1}^{\infty} U_i^{\text{open}}\right) \cup \left(\bigcap_{i=1}^{\infty} V_i^{\text{open}}\right)$

$$= \bigcap_{i=1}^{\infty} \left(U_i^{\text{open}} \cup V_i^{\text{open}}\right) = \bigcap_{i=1}^{\infty} (U_i \cup V_i)^{\text{open}}.$$

Thus, $G_1 \cup G_2$ is G_δ.

$\therefore$ $E_1 \cup E_2 \in \mathcal{M}$.

Thm11(iii) : primitive defn of mble set

Let $E_1, E_2 \in \mathcal{M}$. $\therefore \exists\ F_1, F_2\ (F_\sigma) \ni F_i \subseteq E_i$ and $m^*(E_i \sim F_i) = 0$ $[i \in \{1,2\}]$.

$$0 = m^*(E_1 \sim F_1) + m^*(E_2 \sim F_2) \geq m^*((E_1 \sim F_1) \cup (E_2 \sim F_2)) = m^*(((E_1 \cup E_2) \sim (F_1 \cup F_2)) \cup (E_1 \cap F_2) \cup (E_2 \cap F_1))$$

$$\geq m^*((E_1 \cup E_2) \sim (F_1 \cup F_2)) \geq 0 \quad \Rightarrow \quad m^*((E_1 \cup E_2) \sim (F_1 \cup F_2)) = 0.$$

Now, $F_1 = \bigcup_{i=1}^{\infty} U_i^{\text{closed}}$, $F_2 = \bigcup_{i=1}^{\infty} V_i^{\text{closed}}$ $\Rightarrow F_1 \cup F_2 = \bigcup_{i=1}^{\infty} (U_i \cup V_i)^{\text{closed}}$ $\in$ F_σ.

$\therefore$ $E_1 \cup E_2 \in \mathcal{M}$.

22. $m^{**}(A) \in [0,\infty]$; $m^{**}(A) = \inf\{m^*(\mathcal{O}) \mid \mathcal{O} \supseteq A, \mathcal{O} \text{ open}\}$.

How is m^{**} related to m^*?

Soln. $A \subseteq \mathbb{R}$.

$\{I_k\}_{k=1}^{\infty}$ ↑ bdd, open intervals

$$\sum_{k=1}^{\infty} \ell(I_k) \geq m^*\left(\bigcup_{k=1}^{\infty} I_k\right) \geq m^{**}(A) \Rightarrow m^*(A) \geq m^{**}(A).$$

$m^{**}(A) = \infty \Rightarrow m^{**}(A) \geq m^*(A)$.

Suppose $m^{**}(A) < \infty$. $\forall \varepsilon, \exists\, \mathcal{O}^{\text{open}} \ni A \subseteq \mathcal{O}$ and $m^{**}(A) > m^*(\mathcal{O}) - \varepsilon \geq m^*(A) - \varepsilon$. holds $\forall$ $\varepsilon \Rightarrow m^{**}(A) \geq m^*(A)$

$\therefore\ m^*(A) = m^{**}(A)$.

23. $m^{***}(A) \in [0,\infty]$; $m^{***}(A) = \sup\{m^*(F) \mid F^{\text{closed}} \subseteq A\}$

~~Soln. $m^{***}(A) \geq m^*(F) \geq m^*(A)$ $\forall$ F^{closed}~~

Soln. ~~A bdd~~ Claim 1: A bdd. Then $m^{***}(A) = m^*(A)$ iff $A \in \mathcal{M}$.

↳ $m^{***}(A) = m^*(A)$.

For any $\varepsilon > 0$, $\exists\, F^{\text{closed}} \subseteq A \ni m^{***}(A) < m^*(F) + \varepsilon$.

Excision $\Rightarrow m^*(A \sim F) = m^*(A) - m^*(F) = m^{***}(A) - m^*(F) < \varepsilon$ (arbitrary.)

$\therefore A \in \mathcal{M}$.

$A \in \mathcal{M}$

Fix $\varepsilon > 0$. $\exists\, F^{\text{closed}} \subseteq A \ni m^*(A \sim F) < \varepsilon$. Excision $\Rightarrow m^*(A) < m^*(F) + \varepsilon \leq m^{***}(A) + \varepsilon$.

holds $\forall$ ε $\therefore m^*(A) \leq m^{***}(A)$.

$\because m^*(F) \leq m^*(A)$ $\forall$ $F \subseteq A$, $\therefore m^{***}(A) \leq m^*(A)$.

$\therefore m^*(A) = m^{***}(A)$.

Claim 2. A unbdd ; $A \subseteq \mathbb{R}$. Then $m^{***}(A \cap I) = m^*(A \cap I)$ $\forall$ I bdd interval iff $A \in \mathcal{M}$.

Pf. $m^{***}(A \cap I) = m^*(A \cap I)$ $\forall$ I bdd interval.

$I = (-n, n)$. Claim 1 $\Rightarrow A \cap I_n \in \mathcal{M}$ $\forall n$.

$\therefore A = \bigcup_{n=1}^{\infty} (A \cap I_n) \in \mathcal{M}$.

Conversely, $A \in \mathcal{M}$. Pick bdd interval I. Prop 8 $\Rightarrow I \in \mathcal{M}$.

$\therefore A \cap I \in \mathcal{M}$. Claim 1 $\Rightarrow m^{***}(A \cap I) = m^*(A \cap I)$.

24. $E_1, E_2 \in \mathcal{M}$; p.t. $m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2)$.

Soln. $m(E_1 \cup E_2) - m(E_1) = m((E_1 \cup E_2) \sim E_1) = m(E_2 \sim E_1) = m(E_2 \sim (E_1 \cap E_2))$

$\because E_1, E_1 \cup E_2 \in \mathcal{M}$ $\qquad = m(E_2) - m(E_1 \cap E_2)$ $\quad [\because E_1 \cap E_2 \in \mathcal{M}]$.

Let $m(E_1), m(E_2) < \infty$.

otherwise, trivial

25. Show that the assumption $m(B_1) < \infty$ is necessary in part (ii) Continuity thm.

Soln. ~~Let $m(B_1) = \infty$ and $B_i = B_j$ $\forall$ $B_k = B_{k+1}$ $\forall k \in \mathbb{N}$. Then $\{B_k\}_{k=1}^{\infty}$ is indeed descending~~

Consider $\{B_k\}_{k=1}^{\infty}$; $B_k = (k, \infty)$ Then $m(B_k) = \infty$ $\forall k \in \mathbb{N}$ and $\bigcap_{k=1}^{\infty} B_k = \phi$.

$\Rightarrow 0 = m\left(\bigcap_{k=1}^{\infty} B_k\right) \neq \lim_{k \to \infty} m(B_k) = \infty$.

26. $\{E_k\}_{k=1}^{\infty} \subseteq \mathcal{M}$. $E_i \cap E_j = \phi$ $\forall i \neq j$. P.t. for any set A, $m^*\left(A \cap \bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} m^*(A \cap E_k)$.

Soln. Countable subadditivity of m^*, $m^*\left(A \cap \bigcup_{k=1}^{\infty} E_k\right) = m^*\left(\bigcup_{k=1}^{\infty} (A \cap E_k)\right) \leq \sum_{k=1}^{\infty} m^*(A \cap E_k)$.

- $m^*\left(A \cap \bigcup_{k=1}^{\infty} E_k\right) = \infty$, $\Rightarrow \sum_{k=1}^{\infty} m^*(A \cap E_k) \leq m^*\left(A \cap \bigcup_{k=1}^{\infty} E_k\right)$
- $m^*\left(A \cap \bigcup_{k=1}^{\infty} E_k\right) < \infty$. (Prob 7) find G_δ set $G \ni A \cap \bigcup_{k=1}^{\infty} E_k \subseteq G$ and $m^*(G) = m^*\left(A \cap \bigcup_{k=1}^{\infty} E_k\right)$.

$\because G \in \mathcal{M}$, $\{G \cap E_k\}_{k=1}^{\infty} \subseteq \mathcal{M}$; $(G \cap E_i) \cap (G \cap E_j) = \phi$ if $i \neq j$.

$$\therefore \quad m^*\left(A \cap \bigcup_{k=1}^{\infty} E_k\right) = m^*(G) \geq m^*\left(G \cap \bigcup_{k=1}^{\infty} E_k\right) = m^*\left(\bigcup_{k=1}^{\infty} (G \cap E_k)\right) = \sum_{k=1}^{\infty} m^*(G \cap E_k) \geq \sum_{k=1}^{\infty} m^*(A \cap E_k)$$

●

27. $\mathcal{M}'$ be any σ-algebra of subsets of $\mathbb{R}$; $m' : \mathcal{M}' \to [0, \infty]$
$\downarrow$
countably additive, $m'(\phi) = 0$

(i) S.t. m' finitely additive, monotone, countably monotone, excision.

(ii) S.t. m' possesses the same continuity properties as Lebesgue measure.

<u>Soln</u> (i) $m' : \mathcal{M}' \to [0, \infty]$ countably additive

if $\{E_k\}_{k=1}^{\infty} \subseteq \mathcal{M}'$ $\ni$ $E_i \cap E_j = \phi$ if $i \neq j$ then $m'\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} m'(E_k)$

~~Take E_k~~ Given $\{B_k\}_{k=1}^{n}$. Define $B_l = \phi$ for $k+1 \leq l \in \mathbb{N}$. $\because m'(\phi) = 0$,

$$m'\left(\bigcup_{k=1}^{n} B_k\right) = \sum_{k=1}^{n} m'(B_k).$$

<u>Monotonicity</u> given $A \subseteq B$; $A, B \subseteq \mathcal{M}'$, then $B \sim A \in \mathcal{M}'$ (σ-algebra)

$$\underbrace{m^*(B \sim A)}_{\geq 0} + m^*(A) = m^*(B) \Rightarrow m^*(A) \leq m^*(B)$$

<u>Countable monotonicity</u>: Let $E = \bigcup_{k=1}^{\infty} E_k$, then $E \in \mathcal{M}$, $m^*(E) = \sum_{k=1}^{\infty} m^*\left(E_k \setminus \bigcup_{i=1}^{k-1} E_i\right)$

$$= \bigcup_{k=1}^{\infty} \Big\{ \underbrace{E_k \setminus \Big(\bigcup_{i=1}^{k-1} E_i \Big)}_{\in \mathcal{M}} \Big\}$$

$$\leq \sum_{k=1}^{\infty} m^*(E_k) \text{ by monotonicity}$$

<u>Excision</u> (If $B \notin \mathcal{M}'$ $\nexists$, I don't know what to do)

(ii) <u>T.S.T.</u>: 1. $\{A_k\}_{k=1}^{\infty} \subseteq \mathcal{M}'$, $A_k \subseteq A_{k+1}$ $\forall k \in \mathbb{N}$ then $m'\left(\bigcup_{k=1}^{\infty} A_k\right) = \lim_{n \to \infty} m(A_k)$

2. $\{B_k\}_{k=1}^{\infty} \subseteq \mathcal{M}'$, $B_{k+1} \subseteq B_k$ $\forall k \in \mathbb{N}$ then $m'\left(\bigcap_{k=1}^{\infty} B_k\right) = \lim_{k \to \infty} m'(B_k)$ if $m'(B_1) < \infty$.

1. Case I $\exists k_0 \in \mathbb{N} \ni m'(A_{k_0}) = \infty$; by monotonicity $m'\left(\bigcup_{k=1}^{\infty} A_k\right) = \infty$ and $m'(A_k) = \infty \ \forall \ k \geq k_0$.

Case II ($m'(A_k) < \infty \ \forall \ k \in \mathbb{N}$) $A_0 := \phi$, $C_k = A_k \sim A_{k-1} \ \forall \ k \geq 1$ $\{A_k\}_{k=1}^{\infty} \uparrow$. $\therefore C_i \neq C_j$ if $i \neq j$

$\bigcup_{k=1}^{\infty} A_k = \bigcup_{k=1}^{\infty} C_k$. By countable additivity of m'

$$m^{*\prime}\left(\bigcup_{k=1}^{\infty} A_k\right) = m'\left(\bigcup_{k=1}^{\infty} C_k\right) = \sum_{k=1}^{\infty} m'(A_k \sim A_{k-1}).$$

$\because \{A_k\}_{k=1}^{\infty} \uparrow$. $\therefore \sum_{k=1}^{\infty} m'(A_k \sim A_{k-1}) = \sum_{k=1}^{\infty} [m'(A_k) - m'(A_{k-1})] = \lim_{n\to\infty} \sum_{k=1}^{n} [m'(A_k) - m'(A_{k-1})]$

$= \lim_{n\to\infty} [m'(A_n) - m'(A_0)]$.

$m'(A_0) = m'(\phi) = 0$.

2. $D_k = B_1 \sim B_k \ \forall \ k \in \mathbb{N}$. $\{B_k\}_{k=1}^{\infty} \downarrow \Rightarrow \{D_k\}_{k=1}^{\infty} \uparrow$.

(i) $\Rightarrow m'\left(\bigcup_{k=1}^{\infty} D_k\right) = \lim_{k\to\infty} m'(D_k)$.

De Morgan $\Rightarrow \bigcup_{k=1}^{\infty} D_k = \bigcup_{k=1}^{\infty} [B_1 \sim B_k] = B_1 \sim \bigcap_{k=1}^{\infty} B_k$.

Excision $\Rightarrow m'(D_k) = m'(B_1) - m'(B_k)$ $[\because m'(B_k) < \infty]$ $\forall \ k \in \mathbb{N}$.

$\therefore m'\left(B_1 \sim \bigcap_{k=1}^{\infty} B_k\right) = \lim_{n\to\infty} [m'(B_1) - m'(B_n)]$

$\parallel$

$m'(B_1) - m'\left(\bigcap_{k=1}^{\infty} B_k\right)$. by excision

28. Show that continuity of measure together with finite additivity of measure implies countable additivity of measure.

Soln. $\mathcal{M}'$: σ-algebra of subsets of $\mathbb{R}$

m', set function : $\mathcal{M}' \to [0, \infty]$.

$\downarrow$

finitely additive ; $\underset{\subseteq \mathcal{M}}{\{M_k\}_{k=1}^{\infty}} \uparrow \Rightarrow m'\left(\bigcup_{k=1}^{\infty} M_k\right) = \lim_{k\to\infty} m'(M_k)$.

$\{M_k\}_{k=1}^{\infty} \subseteq \mathcal{M}'$; ~~$M_i \neq M_j$~~ $M_i \cap M_j = \phi$ if $i \neq j$.

$\tilde{M}_k = \bigcup_{i=1}^{k} M_i$. Then $\underbrace{\{\tilde{M}_k\}_{k=1}^{\infty}}_{\subseteq \mathcal{M}'} \uparrow$ and $\bigcup_{k=1}^{\infty} M_k = \bigcup_{k=1}^{\infty} \tilde{M}_k$.

$$\therefore m'\left(\bigcup_{k=1}^{\infty} M_k\right) = m'\left(\bigcup_{k=1}^{\infty} \tilde{M}_k\right) = \underbrace{\lim_{k\to\infty} m'(\tilde{M}_k)}_{\text{continuity}} = \underbrace{\lim_{k\to\infty} \sum_{i=1}^{k} m'(M_i)}_{\text{finite additivity}} = \sum_{k=1}^{\infty} m'(M_k)$$

29. (i) Show that rational equivalence defines an equivalence relation on any set.

(ii) Explicitly find a choice set for the rational equivalence relation on $\mathbb{Q}$.

(iii) Define 2 nos. to be irrationally equivalent provided their difference $\in \mathbb{R}\setminus\mathbb{Q}$.

Is it an equivalence rln on $\mathbb{R}$? Is this an equivalence rln on $\mathbb{Q}$?

<u>Soln.</u> (i) Let $E \subseteq \mathbb{R}$. ~~Let~~ $e_1, e_2 \in E$; $e_1 \sim_{\mathbb{Q}} e_2 \Leftrightarrow e_1 - e_2 \in \mathbb{Q}$.

<u>Symmetry</u> $e_1 \sim_{\mathbb{Q}} e_2 \Rightarrow e_1 - e_2 \in \mathbb{Q} \Rightarrow e_2 - e_1 \in \mathbb{Q} \Rightarrow e_2 \sim_{\mathbb{Q}} e_1$

<u>Reflexivity</u> $e_1 \sim_{\mathbb{Q}} e_1 \because e_1 - e_1 = 0 \in \mathbb{Q}$

<u>Transitivity</u> $e_1 \sim_{\mathbb{Q}} e_2,\ e_2 \sim_{\mathbb{Q}} e_3 \Rightarrow e_1 - e_2,\ e_2 - e_3 \in \mathbb{Q} \Rightarrow (e_1 - e_2) + (e_2 - e_3) \in \mathbb{Q} \Rightarrow e_1 - e_3 \in \mathbb{Q}$
$\Downarrow$
$e_1 \sim_{\mathbb{Q}} e_3$.

(ii) Since the difference between any 2 rational nos. is rational, all elts of $\mathbb{Q}$ $\in$ a single equivalence class

$\{q\}$ for any $q \in \mathbb{Q}$

(iii) <u>Not equivalence rln on $\mathbb{R}$.</u> $\alpha_1, \alpha_2 \in \mathbb{R} \ni \alpha_1 - \alpha_2 \in \mathbb{R}\setminus\mathbb{Q}$

$\alpha_2, \alpha_3 \in \mathbb{R} \ni \alpha_2 - \alpha_3 = -(\alpha_1 - \alpha_2) \in \mathbb{R}\setminus\mathbb{Q}$.

But $\alpha_1 - \alpha_3 = (\alpha_1 - \alpha_2) + (\alpha_2 - \alpha_3) = (\alpha_1 - \alpha_2) - (\alpha_1 - \alpha_2) = 0 \in \mathbb{Q}$

$\therefore$ Not transitive.

~~Equally~~

Not equivalence on $\mathbb{Q}$: $a \in \mathbb{Q}$. $a - a = 0 \notin \mathbb{R} \setminus \mathbb{Q}$. $\therefore$ Not reflexive.

30. S.t. any choice set for rational equivalence rln on a set of +ve outer measure must be uncountably infinite.

Soln. $\phi \neq E \subseteq \mathbb{R}$ $\quad m^*(E) > 0$

$\mathcal{C}_E$ choice set for $\sim_{\mathbb{Q}}$ on E.

$x \in \mathcal{C}_E$, $E_x = \langle x \rangle$

$E_x \subseteq \{x + q \cdot q \in \mathbb{Q}\}$ $\forall x \in \mathcal{C}_E$. $\mathbb{Q}$ countable $\Rightarrow E_x$ countable.

. $\mathcal{C}_E$ countable $\Rightarrow$ $E = \bigcup_{x \in \mathcal{C}_E} E_x$ countable $\Rightarrow m^*(E) = 0$ $\rightarrow\leftarrow$

31. Justify the assertion in the proof of Vitali's thm that it suffices to consider the case that E is bdd.

Soln. If not, it contains a subset of finite outer measure, whose subset we want to construct.

32. $|\Lambda| < \infty$ or Λ uncountably finite or Λ unbdd. Is Lem 16 true?

Soln. $E = [0, 1]$ (counterexample)

$\Lambda = \{2, 4, 6, 8\}$; $|\Lambda| < \infty$

$\{\lambda + E\}_{\lambda \in \Lambda} = \{2 + [0,1],\ 4 + [0,1],\ 6 + [0,1],\ 8 + [0,1]\} = \{[2,3], [4,5], [6,7], [8,9]\}$

disjoint

$m(E) = 1$ ..

Λ unbdd (counterexample) $\quad E = [0,1]$

$\Lambda = \{2, 4, 6, 8, 10, \dots\} = 2\mathbb{Z}^+$; countably infinite.

$\{\lambda + E\}_{\lambda \in \Lambda} = \{[2k, 2k+1] \mid k \in \mathbb{N}\}$ disjoint $\quad m(E) = 1$.

If Λ is uncountably infinite and bounded, we can find a countable subset of Λ that satisfies the conditions of the lemma. Thus conclusion still holds.

33. $E \notin \mathcal{M}$; $m^*(E) < \infty$. s.t. $\exists$ G_δ set $G \ni E \subseteq G$ and $m^*(E) = m^*(G)$ while $m^*(G \sim E) > 0$.

<u>Soln.</u> Prob 7 $\Rightarrow \exists$ G_δ set $G \ni E \subseteq G$ & $m^*(G) = m^*(E)$.

$m^*(G \sim E) = 0 \Rightarrow G \sim E \in \mathcal{M} \Rightarrow E = G \cap (G \sim E)^c \in \mathcal{M}$. $\rightarrow\leftarrow$

34. Show that $\exists$ a continuous, strictly increasing fn on $[0,1]$ that maps a set of positive measure onto a set of measure 0.

<u>Soln.</u> $f(x) = \dfrac{\psi(x)}{2}$; $\psi : [0,1] \to [0,2]$

f cts, $f \uparrow$ composition of 2 cts, strictly increasing functions

Range $f = [0,1]$.

$\mathcal{O} = [0,1] \sim \mathbb{C}$; $f(\mathcal{O}) \cap f(\mathbb{C}) = \phi$; $m(f(\mathcal{O})) = \frac{1}{2}$.

$[0,1] = f([0,1]) = f(\mathcal{O}) \cup f(\mathbb{C}) \Rightarrow m(f(\mathbb{C})) = \frac{1}{2}$

$f^{-1} : [0,1] \to [0,1]$; cts, $\Uparrow$

$f(\mathbb{C}) \mapsto \mathbb{C}$

$m(f(\mathbb{C})) > 0$ $m(\mathbb{C}) = 0$.

35. $f \uparrow$ on I open interval, $x_0 \in I$. Show that f cts @ x_0 iff $\exists$ $\{a_n\}_{n\in\mathbb{N}}, \{b_n\}_{n\in\mathbb{N}}$ in $I \ni \forall n \in \mathbb{N}$, $a_n < x_0 < b_n$ and $\lim_{n\to\infty} (f(b_n) - f(a_n)) = 0$.

<u>Soln.</u>

($\Leftarrow$) f cts at $x_0 \in I$.

I open $\Rightarrow \exists$ $r > 0 \ni (x_0 - r, x_0 + r) \subseteq I$.

Pick sequences $b_n \in (x_0, x_0 + r)$, $a_n \in (x_0 - r, x_0)$ that converge to x_0.

$a_n < x_0 < b_n$; $\{a_n\}, \{b_n\} \in I$; $\lim_{n\to\infty} (f(b_n) - f(a_n)) = 0$.

~~Linearity~~

(Linearity and Monotonicity of convergence of real sequences) ~~for~~

$\{a_n\}, \{b_n\}$ convergent sequences of real nos.

$\forall\ \alpha, \beta \in \mathbb{R}$, $\{\alpha \cdot a_n + \beta \cdot b_n\}_{n \in \mathbb{N}}$ is convergent and

$$\lim_{n\to\infty} (\alpha \cdot a_n + \beta \cdot b_n) = \alpha \cdot \lim_{n\to\infty} a_n + \beta \cdot \lim_{n\to\infty} b_n$$

$$a_n \leq b_n \ \forall\, n \in \mathbb{N} \Rightarrow \lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n .$$

($\Rightarrow$) $\{a_n\}, \{b_n\}$ sequences in I.

$\forall\ n \in \mathbb{N}$, $a_n < x_0 < b_n$ and $\lim_{n\to\infty} \left[f(b_n) - f(a_n)\right] = 0$.

fix $\varepsilon > 0$; choose $n \ni |f(b_n) - f(a_n)| < \varepsilon$.

$$\delta = \min\{x_0 - a_n, b_n - x_0\}.$$

$$|x - x_0| < \delta \Rightarrow a_n < x_0 - \delta < x < x_0 + \delta < b_n \Rightarrow f(a_n) \leq f(x) \leq f(b_n) \quad [\because f \uparrow]$$

$\therefore\ f(a_n) - f(b_n) \leq f(x) - f(x_0) \leq f(b_n) - f(a_n)$.

$\Rightarrow |f(x) - f(x_0)| \leq |f(b_n) - f(a_n)| < \varepsilon$.

36. Show that if $f : [0,1] \to \mathbb{R}$, $f \uparrow$, $f(x) = \varphi(x)\ \forall\, x \in \mathcal{O}$ then $f = \varphi$ on all of $[0,1]$.

Soln. $\mathcal{O} = [0,1] \sim \mathcal{C}$.

fix $x \in (0,1) \cap \mathcal{C}$. Then $\varphi(t) = f(t) \leq f(x)\ \forall\, t \in \mathcal{O} \cap [0, x)$.

$\therefore\ \varphi(x) \leq f(x)$.

fix $\varepsilon > 0$. φ cts $\Rightarrow \exists\ \delta > 0 \ni \varphi(t) < \varphi(x) + \varepsilon\ \ \forall\, t \in [0,1] \ni |x - t| < \delta$.

$m(\mathcal{C}) = 0 \Rightarrow (x, x+\delta) \not\subseteq \mathcal{C}$. $\therefore \exists\ t \in \mathcal{O} \cap (x, x+\delta) \ni f(x) \leq f(t) = \varphi(t) < \varphi(x) + \varepsilon$.

$\because f(x) \leq \varphi(x) + \varepsilon$ for arbitrary $\varepsilon > 0$, $f(x) \leq \varphi(x)$.

Note that under the given assumptions, f may not agree with φ at 0 or 1.

37. $f^{\text{cts}}: E \to \mathbb{R}$. Is it true that $\mathbb{R} \supseteq A \in \mathcal{M} \Rightarrow f^{-1}(A) \in \mathcal{M}$?

<u>Soln.</u> Prop 21. ~~We~~ $W \subseteq \psi(C)$; $W \notin \mathcal{M}$. ψ cts, 1-1

$$\Downarrow$$

$$\psi^{-1} \text{ is cts}$$

$$\psi^{-1}: \underset{\notin \mathcal{M}}{W} \to \psi^{-1}(W) \in \mathcal{M}.$$

The pre-image of a measurable set need not be measurable.

38. $f: [a,b] \to \mathbb{R}$

$\downarrow$

Lipschitz, i.e., $\exists$ constant $c \geq 0 \ni \forall u, v \in [a,b]$, $|f(u) - f(v)| \leq c|u-v|$

S.t. f maps a set of measure zero onto a set of measure zero.

f maps an F_σ set onto an F_σ set.

Conclude that f maps a measurable set onto a measurable set.

<u>Soln.</u> $\phi \neq E \subseteq [a,b]$, $\downarrow$ $m(E) = 0$ | fix $\varepsilon > 0$.

$\exists\, \mathcal{O}^{\text{open}} \ni E \subseteq \mathcal{O}$ & $m(\mathcal{O}) = m(\mathcal{O} \sim E) < \frac{\varepsilon}{c}$ $\leftarrow$ Lipschitz of c

$\mathcal{O} = \{I_k^{\text{~~disjoint~~ open interval}}\}_{k=1}^{\infty}$; $I_{k_1} \cap I_{k_2} = \phi$ if $k_1 \neq k_2$.

$\forall I_k$ $(k \in \mathbb{N})$, $I_k' := I_k \cap [a,b]$, $a_k = \inf\limits_{u \in I_k'} f(u)$ and $b_k = \cancel{\inf} \sup\limits_{u \in I_k'} f(u)$.

$$\underbrace{m(f(I_k')) \leq b_k - a_k}_{f(I_k') \subseteq (a_k, b_k)} \leq \sup_{u,v \in I_k'} |f(u) - f(v)| \leq c \sup_{u,v \in I_k'} |u - v| < c\,\ell(I_k).$$

$$\therefore m(f(E)) \leq m(f(\mathcal{O} \cap [a,b])) = m\left(f\left(\bigcup_{k=1}^{\infty} I_k'\right)\right) = m\left(\bigcup_{k=1}^{\infty} f(I_k')\right) \leq \sum_{k=1}^{\infty} m(f(I_k'))$$

$$\leq c \sum_{k=1}^{\infty} \ell(I_k) = c\, m(\mathcal{O}) < \varepsilon.$$

$$\ldots \forall \varepsilon > 0 \Rightarrow m(f(E)) = 0.$$

Claim: A cts image of a closed and bounded set of real nos. is also closed and bdd.

Pf of claim: $A^{\text{closed, bdd}} \subseteq \mathbb{R}$ | $\{E_k\}_{k=1}^{\infty}$ open cover of $f(A)$.

$f^{\text{cts}} : A \longrightarrow \mathbb{R}$ | $f^{-1}(E_k) = A \cap \underbrace{\mathcal{U}_k}_{\text{open}}$

$$\bigcup_{k=1}^{\infty} f^{-1}(E_k) = f^{-1}\left(\bigcup_{k=1}^{\infty} E_k\right) \supseteq f^{-1}(f(A)) \supseteq A.$$

$\therefore \{\mathcal{U}_k\}_{k=1}^{\infty}$ open cover of A $\xrightarrow{\text{Hein-Borel}}$ $\{\mathcal{U}_k\}_{k=1}^{N}$

$$f(A) = f\left(\bigcup_{k=1}^{N}(A \cap \mathcal{U}_k)\right) = f\left(f^{-1}\left(\bigcup_{k=1}^{N} E_k\right)\right) \subseteq \bigcup_{k=1}^{N} E_k.$$

$$\therefore f(A) \subseteq \bigcup_{k=1}^{N} E_k.$$

$\therefore f(A)$ compact $\Rightarrow$ closed and bdd.

Back to soln. $\boxed{F_\sigma}$ $E \subseteq [a,b]$. $\therefore \exists \{F_k^{\text{closed}}\}_{k=1}^{\infty} \ni E = \bigcup_{k=1}^{\infty} F_k$

$\downarrow$

F_σ Lipschitz fns are cts $\Rightarrow$ $f(F_k)$ closed and bdd $\forall\, k \in \mathbb{N}$.

$$f(E) = \bigcup_{k=1}^{\infty} f(F_k), \quad F_\sigma.$$

$\boxed{\mathcal{M}}$ $E \subseteq [a,b] \ni E \in \mathcal{M}$.

$\exists\, F_\sigma$ set $F \subseteq E \ni m(E \sim F) = 0$.

$$f(E) = \underset{\in \mathcal{M}}{f(F)} \cup \underset{\in \mathcal{M}}{f(E \sim F)} \Rightarrow f(E) \in \mathcal{M}.$$

39. $F \subseteq [0,1]$

$\downarrow$

constructed in the same manner as the Cantor set except that each of the intervals removed at the n^{th} deletion stage has length $\alpha 3^{-n}$ with $0 < \alpha < 1$. s.t. F is a closed set, $[0,1] \sim F$ dense in $[0,1]$, $m(F) = 1 - \alpha$.

Such a set F is called a generalized Cantor set.

Soln. F_k : set of points that remain after k removal operations

↳ union of 2^k disjoint closed intervals, each of length $\ell_k := 2^{-k}\left(1-\alpha+\alpha\left(\frac{2}{3}\right)^k\right)$.

finite union of closed sets is closed $\Rightarrow$ F_k closed.

↳ intersection of closed sets is closed

$$F = \bigcap_{k=1}^{\infty} F_k \quad \text{closed.}$$

$\mathcal{O} = [0,1] \sim F \longrightarrow$ open

$x, y \in [0,1]$ $\quad x \in \mathcal{O}$ or $y \in \mathcal{O} \Rightarrow \exists\, x' \ni x < x' < y$ and $x' \in \mathcal{O}$.

Suppose $x, y \in F$

Choose $k \in \mathbb{N} \ni \ell_k < |x-y|$

$$F_k = \bigcup_{n=1}^{2^k} I_n \quad \text{where } I_n \text{ closed interval; } \ell(I_n) = k \ \forall\ n \in \{1, \dots, 2^k\}.$$

$\because$ $x, y \in F_k \Rightarrow x, y \in \bigcup_{n=1}^{2^k} I_n$. x and y cannot belong to the same interval since $\ell_k < |x-y|$.

Since the intervals are disjoint, $\exists\, x' \in (x, y) \ni x' \notin F_k \Rightarrow x' \notin F$.

Thus $\mathcal{O}$ is dense in $[0,1]$.

Observe, $\mathcal{O}$ countable union of disjoint colln of open intervals which are removed during the construction of F.

$\boxed{k^{th} \text{ deletion stage}}$ 2^{k-1} intervals of length $\alpha 3^{-k}$ are removed.

$$\therefore\ \ell(\mathcal{O}) = m(\mathcal{O}) = \frac{\alpha}{2}\sum_{k=1}^{\infty}\left(\frac{2}{3}\right)^k = \alpha \Rightarrow m(F) = 1-\alpha \text{ by excision.}$$

40. Show that there is an open set of real nos. that, contrary to intuition, has a boundary of +ve measure.

Soln. F : generalized cantor set. $m(F) = 1-\alpha$. $\mathcal{O} = [0,1] \sim F$, open

$\mathcal{O}$ does not contain any of its boundary points. But since $\mathcal{O}$ is dense in $[0,1]$, every point in F is a boundary point of O.

$\therefore \partial O = F$; $m(\partial O) = m(F) = 1-\alpha$.

41. __Defn.__ $\phi \neq X \subseteq \mathbb{R}$
$\downarrow$
perfect if (i) X, closed

(ii) $x \in X$, $\Rightarrow |N(x) \cap X| = \infty$ where $N(x)$ is any nbd of X

S.t. the Cantor set is perfect.

__Soln.__ $x \in C$; fixed $\varepsilon > 0$.

Choose $k \ni 3^{-k} < \varepsilon$.

C_k : points that remain after k removal operations

$\hookrightarrow$ union of a pairwise disjoint colln of 2^k closed intervals, each of length 3^{-k}.

$x \in C_k \Rightarrow \exists$ interval $I = [a, b]$ in this colln that contains x.

The endpoints of the intervals ~~are~~ in the colln are never removed in the constn of the Cantor set, so $a, b \in C$. Furthermore both $a, b \in (x-\varepsilon, x+\varepsilon)$ since $I \subseteq (x-\varepsilon, x+\varepsilon)$. But x cannot equal both a and b, so one of the endpoints is in the set $C \cap (x-\varepsilon, x+\varepsilon) \setminus \{x\}$.

This process can be repeated $\forall\ k' > k$ to generate an infinite colln of points in $C \cap (x-\varepsilon, x+\varepsilon)$. Since ε was arbitrary and C closed, C is perfect.

The endpoints of all the subintervals occurring in the Cantor construction belong to C.

---

42. P.t. every perfect $X \subseteq \mathbb{R}$ is uncountable.

__Soln.__ Suppose X countable.

$\{x_k\}_{k=1}^{\infty}$ enumeration of X.

$n_1 = 1$, $U_1 = (x_{n_1} - 1, x_{n_1} + 1)$. X perfect set $\Rightarrow |U_1 \cap X| = \infty$.

$n_2 = \min \{k \quad x_k \in U_1 \sim \{x_{n_1}\}\}$

U_2 be an open interval satisfying $x_{n_2} \in U_2$

$x_{n_1} \notin \bar{U}_2$

$\bar{U}_2 \subset U_1$

By continuing in this fashion, we obtain $\{U_n\}_{n=1}^{\infty}$

$x_k \notin \bar{U}_{n+1}$ for $k = 1, \ldots, n$.

$U_n \cap X \neq \phi$ for $n \in \mathbb{N}$

$\bar{U}_{n+1} \subset U_n$ for $n \in \mathbb{N}$.

$E_n = \bar{U}_n \cap X$. $\{E_n\}_{n=1}^{\infty} \downarrow$, $E_n \neq \phi \ \forall n \in \mathbb{N}$.

E_1 bdd.

Nested set thm $\Rightarrow \bigcap_{n=1}^{\infty} E_n \neq \phi$.

$\bigcap_{n=1}^{\infty} E_n \subseteq X$, $x_k \in \bigcap_{n=1}^{\infty} E_n$ for some k.

$x_k \in \bar{U}_{k+1}$ $\rightarrow\leftarrow$

If X is countable, construct a descending sequence of bounded, closed subsets of X whose intersection is empty.

43. Another proof of the uncountability of the Cantor set. (41 & 42)

44. $A \subseteq \mathbb{R}$
↳ nowhere dense in $\mathbb{R}$ if every open set O has an open subset disjoint from A.

s.t. $\mathbb{C}$ nowhere dense in $\mathbb{R}$.

Soln. $\mathbb{C}$ cannot contain any open interval. For if we could find an open interval $I \subseteq \mathbb{C}$, then $m(\mathbb{C}) \geq m(I) > 0$ $\rightarrow\leftarrow$

fix interval I.

$\exists\, x \in I \sim \mathcal{C}$. $\mathcal{C}$ closed $\Rightarrow I \sim \mathcal{C}$ open

$\Rightarrow \exists$ interval $I_x \ni x \in I_x$; $I_x \subseteq I \sim \mathcal{C}$.

I_x is an open interval contained in I that is disjoint from $\mathcal{C}$.

Any open set contains an open interval

45. Show that a strictly increasing function that is defined on an interval has a cts inverse.

Soln. A, interval | Since f is strictly monotone, it defines a one-to-one correspondence between the sets A and B.

$f \Uparrow$

$B = f(A)$ | $\therefore f^{-1}\ B \to A$ well-defined.

$y \in B$, fixed.

$$f_y(\alpha) := f(f^{-1}(y) + \alpha) \quad ; \alpha \in A - f^{-1}(y).$$

Note, $f_y(0) = y$, $f_y \Uparrow$ in α and $f_y^{-1}(y') = f^{-1}(y') - f^{-1}(y)$ for $y \in B$.

Fix $\varepsilon > 0$. If $f^{-1}(y) \in \text{int}(A)$, we can find $\varepsilon' \in (0, \varepsilon) \ni -\varepsilon', \varepsilon' \in A - f^{-1}(y)$. Define

$$\delta = \min\{y - f_y(-\varepsilon'), f_y(\varepsilon') - y\}$$

$f^{-1}(y) \in LB(A)$	$f^{-1}(y) \in UB(A)$
choose $\varepsilon' \in (0, \varepsilon) \ni \varepsilon' \in A - f^{-1}(y)$	choose $\varepsilon' \in (0, \varepsilon) \ni -\varepsilon' \in A - f^{-1}(y)$
$\delta := f_y(\varepsilon') - y$	$\delta := y - f_y(-\varepsilon')$.
Pick $y' \in B \ni \lvert y - y'\rvert < \delta$. If $y' \geq y$ then $f_y(f_y^{-1}(y')) - y = \lvert y - y'\rvert < \delta \leq f_y(\varepsilon') - y \Rightarrow f_y^{-1}(y') < \varepsilon$.	If $y' < y$, $y - f_y(f_y^{-1}(y')) = \lvert y - y'\rvert < \delta \leq y - f_y(-\varepsilon')$ $\Rightarrow f_y^{-1}(y') > -\varepsilon$.

$$\therefore \lvert f_y^{-1}(y')\rvert = \lvert f^{-1}(y') - f^{-1}(y)\rvert < \varepsilon.$$

46. f cts, $B \in \mathcal{B}$.

S.t. $f^{-1}(B) \in \mathcal{B}$.

Soln. $\mathcal{F} := \{E \subseteq \mathbb{R} : f^{-1}(E) \in \mathcal{B}\}$

O, open set.

$\Rightarrow f^{-1}(O)$ open, $f^{-1}(O) \in \mathcal{B}$. $\quad \therefore O \in \mathcal{F}$.

- $f^{-1}(\phi) = \phi \in \mathcal{B} \Rightarrow \phi \in \mathcal{F}$.
- $E \in \mathcal{F} \Rightarrow f^{-1}(E^c) = f^{-1}(E)^c \in \mathcal{B}$, since $f^{-1}(E) \in \underset{\substack{\uparrow \\ \text{closed under complements}}}{\mathcal{B}} \Rightarrow E^c \in \mathcal{F}$.
- $E_k \in \mathcal{F}$ for $k \in \mathbb{N}$.

Then $f^{-1}\left(\bigcup_{k=1}^{\infty} E_k\right) = \bigcup_{k=1}^{\infty} f^{-1}(E_k) \in \mathcal{B}$ $\quad$ [$\because f^{-1}(E_k) \in \underset{\substack{\uparrow \\ \text{closed under countable unions}}}{\mathcal{B}}, \forall k \in \mathbb{N}$]

$\therefore \bigcup_{k=1}^{\infty} E_k \in \mathcal{B}$.

★ $\mathcal{F}$, σ-algebra.

$\mathcal{B}$ smallest σ-algebra containing open sets $\Rightarrow \mathcal{B} \subseteq \mathcal{F}$.

$\Rightarrow f^{-1}(B) \in \mathcal{B} \ \forall \ B \in \mathcal{B}$.

47. (45, 46 $\Rightarrow$) a cts strictly increasing function that is defined on an interval maps Borel sets to Borel sets.

Soln. f cts $\Uparrow$

$\underset{\substack{\uparrow \\ \text{interval}}}{A} \to \mathbb{R}$

$g = f^{-1}$

$\hookrightarrow$ cts

$\therefore$ If B is a Borel set contained in the range of g,

$g^{-1}(B) \in \mathcal{B}$

$g^{-1}(B) = \{y : g(y) \in B\} = \{f(x) : x \in B\} = f(B)$

$\Rightarrow f(B) \in \mathcal{B}$.