Box topology and product topology

Let $X_1, \dots, X_n$ be n topological spaces with respective topologies $\tau_1, \tau_2, \dots, \tau_n$. Then the product set $X_1 \times \dots \times X_n = \{ (x_1, x_2, \dots, x_n) \mid x_i \in X_i \ \forall\, i = 1(1)n \}$ with topology τ given by the basis

$$\mathcal{B} = \{ U_1 \times U_2 \times \dots \times U_n \mid U_i \underset{\text{open}}{\subseteq} X_i \}$$

is called the product topology on $X_1 \times \dots \times X_n$. We have natural projection mappings $\pi_i : \prod_{j=1}^{n} X_j \to X_i$ $(\forall\, i = 1(1)n)$ given by $(x_1, x_2, \dots, x_n) \mapsto x_i$. Let $U_i \subseteq X_i$. Then $\pi_i^{-1}(U_i) = \left(\prod_{\substack{j=1 \\ j \neq i}}^{n} X_i \right) \times U_i$ ignoring the order of the nos. $i = 1(1)n$, with the convention that they appear anyway in their preassigned ascending order. Now, $\left(\prod_{\substack{j=1 \\ j \neq i}}^{n} X_i \right) \times U_i \in \mathcal{B}$ and hence is open for any open set U_i. Thus each projection map is continuous.

We can find a smaller basis for the product topology, for if $\mathcal{E}_i$ be some basis for X_i, then $\mathcal{B}' = \left\{ \prod_{i=1}^{n} C_i \;\middle|\; C_i \in \mathcal{E}_i \ \forall\, i = 1(1)n \right\}$ is another basis for $\prod_{i=1}^{n} X_i$, and is a subset of $\mathcal{B}$.

Note here for the sake of rigour that our naive ignoring of order that made our product representation easy, is valid, since the operation of forming product spaces is commutative and associative.

$$(X \times Y) \times Z \approx X \times (Y \times Z) \qquad X \times Y \approx Y \times X$$

$$((x, y), z) \longleftrightarrow (x, (y, z)) \qquad (x, y) \longleftrightarrow (y, x)$$

and so $X_1 \times \dots \times X_n \approx (X_1 \times \dots \times X_{n-1}) \times X_n$. This is intuitionally true for an infinite no. of spaces as well

Now, a function $f : Z \to X \times Y$ for topological spaces X, Y and Z is continuous iff $\pi_X \circ f : Z \to X$ and $\pi_Y \circ f : Z \to Y$ are in themselves continuous. One way is obvious since projection maps are continuous and composition of continuous functions

is continuous. For the converse, let $p_X \circ f : Z \to X$ is continuous, and so $p_Y \circ f : Z \to Y$ Let $U \times V$ is an open set in $X \times Y$. Then $f^{-1}(U \times V) = (p_X \circ f)^{-1}(U) \cap (p_Y \circ f)^{-1}(V)$ which is the intersection of 2 open sets in Z.

A similar argument would be true for continuous $f : X' \to X$, $g : Y' \to Y$ and $f \times g : X' \times Y' \to X \times Y$. The mapping $f \times g$ is open whenever f and g are open. Given continuous functions $f : Z \to X$ and $g : Z \to Y$, $\exists!$ continuous function $\varphi : Z \to X \times Y$ defined by $\varphi(z) = (f(z), g(z))$. $\varphi \equiv (f, g)$.

For an infinite product of spaces, there are 2 alternatives one can choose to generate a topology on the product space, and, unlike the case of finite product, those two topologies are usually different.

1. Product Topology

Let $\{X_\alpha\}_{\alpha \in A}$ be a collection of spaces with topologies $\{\tau_\alpha\}_{\alpha \in A}$. Choose some $n \in \mathbb{N}$ and pick up n spaces $X_{\alpha_1}, X_{\alpha_2}, \dots, X_{\alpha_n}$. Additionally pick n (proper) open sets $U_{\alpha_1}, \dots, U_{\alpha_n}$ from each of the n chosen spaces. The product topology is given by a basis, which has as elements all such $U_{\alpha_1} \times \dots \times U_{\alpha_n} \times \prod_{\lambda \in A \setminus \{\alpha_1, \dots, \alpha_n\}} X_\lambda$

$$\mathcal{B}_{prod} = \left\{ U_{\alpha_1} \times \dots \times U_{\alpha_n} \times \prod_{\lambda \in A \setminus \{\alpha_1, \dots, \alpha_n\}} X_\lambda \;\middle|\; \begin{array}{l} n \in \mathbb{N},\ \alpha_i \in A \ \ \forall\, i = 1(1) n\,; \\ U_{\alpha_i} \subseteq X_{\alpha_i} \ \ \forall\, i = 1(1) n\,. \end{array} \right\}$$

Let $B_1 = \left(\prod_{i=1}^{n} U_{\alpha_i}\right) \times \left(\prod_{\lambda \in A \setminus \{\alpha_1, \dots, \alpha_n\}} X_\lambda\right)$ and $B_2 = \left(\prod_{i=m}^{N} U_{\alpha_i}\right) \times \left(\prod_{\lambda \in A \setminus \{\alpha_m, \dots, \alpha_N\}} X_\lambda\right)$ be two elements of $\mathcal{B}_{base}$ $(1 \le m \le n \le N)$. To see what their union is we can try seeing what set comes from each topological space in the product:

	$\lambda = \alpha_i$, $i \in \{1, \dots, m\}$	$\lambda = \alpha_i$, $i \in \{m+1, \dots, n\}$	$\lambda = \alpha_i$, $i \in \{n+1, \dots, N\}$	$\lambda \in A \setminus \{\alpha_1, \dots, \alpha_N\}$
space	X_λ	X_λ	X_λ	X_λ
open set in $\mathcal{B}_1$	U_λ	U_λ	X_λ	X_λ
open set in $\mathcal{B}_2$	X_λ	U_λ	U_λ	X_λ
open set in $\mathcal{B}_1 \cup \mathcal{B}_2$	X_λ	U_λ	X_λ	X_λ
open set in $\mathcal{B}_1 \cap \mathcal{B}_2$	U_λ	U_λ	U_λ	X_λ

Thus, we see that $\mathcal{B}_1 \cup \mathcal{B}_2 = \left(\prod_{i=m+1}^{n} U_{\alpha_i}\right) \times \left(\prod_{\lambda \in A \setminus \{\alpha_{m+1}, \dots, \alpha_n\}} X_\lambda\right)$ and so it has proper open subsets only in those entries where $\mathcal{B}_1$ and $\mathcal{B}_2$ both have proper subsets. Thus, in every union of open sets from $\mathcal{B}_{prod}$, it will have proper open subsets in a finite no. of entries, and will have the same form. The topology contains such unions. To see their finite intersections, the last row of the table gives an idea about how the latter type of sets will look like: $\mathcal{B}_1 \cap \mathcal{B}_2 = \left(\prod_{i=1}^{N} U_{\alpha_i}\right) \times \left(\prod_{\lambda \in A \setminus \{\alpha_1, \dots, \alpha_N\}} X_\lambda\right)$

② Box Topology

The box topology is a simple upgradation of the product topology in the finite case, having basis $\mathcal{B}_{box} = \left\{ \prod_{\alpha \in A} U_\alpha \;\middle|\; U_\alpha \underset{\text{open}}{\subseteq} X_\alpha \;\forall\; \alpha \in A \right\}$.

Note that every open set in the product topology is also open in the box topology but not the converse. Thus the product topology is coarser than the box topology over a given collection of sets. Try showing that it is the smallest topology for which each

individual projection map is continuous.

One can rephrase the product topology in the following manner:

(i) Infinite product of sets. One can view $\prod_{\alpha \in A} X_\alpha$ as $\{f : A \to X_\alpha\}$, the set of all functions from the index set to an element of the corresponding participating set.

(ii) Pick some $U_\beta \subseteq_{\text{open}} X_\beta \quad \forall \beta \in A$. $\pi_\beta^{-1}(U_\beta) = U_\beta \times \left(\prod_{\lambda \in A \setminus \{\beta\}} X_\lambda\right)$

$S_\beta = \left\{\pi_\beta^{-1}(U_\beta) \;\middle|\; U_\beta \subseteq_{\text{open}} X_\beta\right\}$ for fixed β.

(iii) $S = \bigcup_{\alpha \in A} S_\alpha$.

Now imagine that S is the subbasis of a topology. Then how will its elements look like? For fixed β, $S_\beta = \left\{U_\beta \times \left(\prod_{\lambda \in A \setminus \{\beta\}} X_\lambda\right) \;\middle|\; U_\beta \subseteq_{\text{open}} X_\beta\right\}$. Then, $\bigcup_{\beta \in A} S_\beta = \bigcup_{\beta \in A} \left\{U_\beta \times \left(\prod_{\lambda \in A \setminus \{\beta\}} X_\lambda\right) \;\middle|\; U_\beta \subseteq_{\text{open}} X_\beta\right\} = \left\{U_\beta \times \prod_{\lambda \in A \setminus \{\beta\}} X_\lambda \;\middle|\; \beta \in A;\ U_\beta \subseteq_{\text{open}} X_\beta\right\}$. Finite intersections of its elements gives elements of the basis. Let $P_1, P_2, \ldots, P_n \in S$; $P_i = U_{\alpha_i} \times \prod_{\lambda \in A \setminus \{\alpha_i\}} X_\lambda \quad \forall i = 1(1)n$.

	$\lambda = \alpha_i$; $i = 1(1)n$				$\lambda \in A \setminus \{\alpha_1, \ldots, \alpha_n\}$
	α_1	α_2	...	α_n	
P_1	U_{α_1}	X_{α_2}		X_{α_n}	X_λ
P_2	X_{α_1}	U_{α_2}		X_{α_n}	X_λ
⋮					
P_n	X_{α_1}	X_{α_2}		U_{α_n}	X_λ

Thus, $\bigcap_{i=1}^{n} P_i = \left(\prod_{i=1}^{n} U_{\alpha_i}\right) \times \left(\prod_{\lambda \in A \setminus \{\alpha_1, \ldots, \alpha_n\}} X_{\alpha \neq \lambda}\right)$ which gives our precious basis elements.

The following are easy to prove:

1. If X_α is Hausdorff $\forall \alpha \in A$, then $\prod_{\alpha \in A} X_\alpha$ is Hausdorff in both box and product

Topologies

2. $\{X_\alpha\}_{\alpha \in A}$ · $A_\alpha \subset X_\alpha \ \forall \ \alpha \in A$. $\prod X_\alpha$ is endowed with either the product or the box topology; then $\prod_{\alpha \in A} \bar{A}_\alpha = \overline{\prod A_\alpha}$ under the same topology.

3. Let X_α $(\alpha \in A)$ be a family of spaces. If $\mathcal{B}_\alpha$ is a basis for the topology of $X_\alpha \ \forall$ $\alpha \in A$ then the collection $\mathcal{C}$ of sets of the form $\prod B_\alpha$ where $B_\alpha = X_\alpha \ \forall$ but finitely many indices α and $B_\alpha \in \mathcal{B}_\alpha$ for the remaining indices, is a basis for the product topology for $\prod X_\alpha$.

4. Preservation of relative topological structure by product topology.

5. Composition with projection : generalize to infinitely many sets.