Ext Functor

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1 Projective Resolution

Let P be a module. For any surjective map $g : B \to C \to 0$. if $f : P \to C$ always has a lifting $h : P \to B$, we say that P is *projective*. Let A be an abelian group. Consider a projective resolution of A

$$P_\# : \ldots \xrightarrow{\partial_{n+2}} P_{n+1} \xrightarrow{\partial_{n+1}} P_n \xrightarrow{\partial_n} P_{n-1} \xrightarrow{\partial_{n-1}} \ldots \xrightarrow{\partial_3} P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\partial_0} A \xrightarrow{\eta} 0,$$

where each P_i is projective and the sequence is exact. Additionally, if each P_i is free, we call $P_\#$ a *free resolution*. For a commutative ring with 1, R, applying the Hom_R functor with an abelian group G, we get a cochain complex that might not be exact:

$$0 \xrightarrow{d_\eta} \mathrm{Hom}_R(A, G) \xrightarrow{d_0} \mathrm{Hom}_R\left(P_0, G\right) \xrightarrow{d_1} \mathrm{Hom}_R\left(P_1, G\right) \xrightarrow{d_2} \ldots$$

The nth cohomology group of this cochain complex gives sort of a measurement of the non-exactness of this cochain complex at the nth position. We define that nth cohomology group as the Ext functor from $\mathcal{A}\mathfrak{b}^2$ to $\mathcal{A}\mathfrak{b}$,

$$\mathrm{Ext}^n_R(A, G) := \ker d_{n+1} / \mathrm{im}\, d_n.$$

It can be proved that this Ext functor does not depend on the particular projective resolution taken, and is just a function of A and G.

If the cochain complex is exact at any point, which will be the case if A is a free abelian group and P a free resolution of A, then $\mathrm{Ext}^n_R(A, G)$ is trivial there.

If A is a free abelian group. Then the free abelian group generated by A is A. Thus we have the chain $0 \to R = \ker \varphi \to \mathcal{F}(A) = A \xrightarrow{\varphi} A \to 0$, where φ ends up being the identity map, and hence $R = 0$. So ∂_0 is just the identity map on A, and ∂_1 takes whole of A to 0; accordingly, $\mathrm{Ext}^0_R(A, G) := \ker d_1 / \mathrm{im}\, d_0$ is trivial. It is also conversely true that a trivial Ext^0 means that the concerned entry abelian group is free. If we consider the exact sequence with the canonical homomorphism $\mathcal{F}(A) \to A$, by assumption it splits which implies that A is projective (because it's a direct summand of the free abelian group $\mathcal{F}(A)$), hence free, abelian group.

2 Examples

Example 1. Let $A = \mathbb{Z}/m\mathbb{Z}$. Consider

$$0 \xrightarrow{\partial_2} \mathbb{Z} \xrightarrow[m]{\partial_1} \mathbb{Z} \xrightarrow{\partial_0} \mathbb{Z}/m\mathbb{Z} \longrightarrow 0$$

This gives us (taking $R = \mathbb{Z}$)

$$0 \longrightarrow \mathrm{Hom}(\mathbb{Z}/m\mathbb{Z}, G) \xrightarrow{d_0} \mathrm{Hom}(\mathbb{Z}, G) \xrightarrow{d_1} \mathrm{Hom}(\mathbb{Z}, G) \xrightarrow{d_2} 0$$

We have the following commutative diagram:

$$\begin{tikzcd}
\overline{m\beta} \in \mathbb{Z}/m\mathbb{Z} \arrow[dd, "\varphi"] & \mathbb{Z} \ni m\beta \arrow[l, "\pi"'] \arrow[ddl, dashed, "\psi"] & \\
& & \mathbb{Z} \ni \beta \arrow[ul, "m"'] \arrow[dll, dashed, "m\psi"] \\
g \in G & &
\end{tikzcd}$$

Here, $d_0(\varphi) = \varphi \circ \pi$, hence im $d_0 = \{$periodic functions with period $m\}$ and ker $d_1 = \{\psi : \mathbb{Z} \to G \mid \psi(m\beta) = 0 \; \forall \, \beta \in \mathbb{Z}\}$. Therefore

$$\mathrm{Ext}^0(\mathbb{Z}/m\mathbb{Z}, G) = \frac{\ker d_1}{\mathrm{im}\; d_0} = \frac{\{\psi : \mathbb{Z} \to G \mid \psi(m\beta) = m\psi(\beta) = 0 \; \forall \, \beta \in \mathbb{Z}\}}{\{\psi : \mathbb{Z} \to G \mid \psi(\alpha + m) = \psi(\alpha) \; \forall \, \alpha \in \mathbb{Z}\}}.$$

Now, for a $[\psi] \in \mathrm{Ext}^0(\mathbb{Z}/m\mathbb{Z}, G)$, $m\psi \equiv 0 \Rightarrow m\psi(\beta) = 0 \; \forall \, \beta \in \mathbb{Z} \Rightarrow m\psi(\beta + m) = 0 \Rightarrow m\psi(\beta) + m\psi(m) = 0$ since ψ is a homomorphism $\Rightarrow m\psi(m) = 0$.

Define $\zeta : \mathrm{Hom}(\mathbb{Z}, G) \to G$; $\psi \mapsto \psi(m) = g \in G$. This is well-defined. Therefore I can identify classes of ψ with corresponding elements of g such that $mg = 0_G$. So, $\mathrm{Ext}^0(\mathbb{Z}/m\mathbb{Z}, G) \approx {}_mG = \{g \in G \mid mg = 0\}$.

Now, let us calculate $\mathrm{Ext}^1(\mathbb{Z}/m\mathbb{Z}, G) = \dfrac{\ker d_2}{\mathrm{im}\; d_1} = \dfrac{\mathrm{Hom}(\mathbb{Z}, G)}{m\mathrm{Hom}(\mathbb{Z}, G)}$.

$$\begin{tikzcd}
\mathbb{Z} \arrow[dr, dashed, "d_1\varphi"] & & \mathbb{Z} \arrow[ll, "m"'] \arrow[dl, "\varphi"] \\
& G &
\end{tikzcd}$$

If $\varphi \in \mathrm{Hom}(\mathbb{Z}, G)$, as soon as we define $\varphi(1)$, φ is fully defined. So, one φ is equivalent to one element $g \in G$ where $\varphi(1)$ is assigned. So, $\mathrm{Hom}(\mathbb{Z}, G) \approx G \Rightarrow Ext^1(\mathbb{Z}/m\mathbb{Z}, G) \approx G/mG$.

Note here that if $|G| \nmid m$, then $\mathrm{Ext}^0(\mathbb{Z}/m\mathbb{Z}, G)$ is non-trivial, providing us with an example of a non-trivial Hom-cochain and Ext, showing that the definition is vacuously true. Furthermore, $\mathrm{Hom}(\mathbb{Z}/m\mathbb{Z}, G) \to \mathrm{Hom}(\mathbb{Z}, G)$ is non-trivial only if G has a torsion.

3 Independence of Ext on resolution

Theorem 1 *Let $A, A' \in \mathcal{A}\mathfrak{b}$. If a homomorphism $f : A \to A'$ exists, it would induce a chain map $f_\#$ in the following scenario:*

$$\begin{tikzcd}
\dots \arrow[r, "\partial_3"] & P_2 \arrow[r, "\partial_2"] \arrow[d, dashed, "f_2"] & P_1 \arrow[r, "\partial_1"] \arrow[d, dashed, "f_1"] & P_0 \arrow[r, "\partial_0"] \arrow[d, dashed, "f_0"] & A \arrow[r, "\eta"] \arrow[d, "f"] & 0 \\
\dots \arrow[r, "\partial_3"] & P'_2 \arrow[r, "\partial'_2"] & P'_1 \arrow[r, "\partial'_1"] & P'_0 \arrow[r, "\partial'_0"] & A' \arrow[r, "\eta'"] & 0
\end{tikzcd}$$

if each P_i in the above row is projective and the bottom row is exact. Moreover, any two such chain maps will be homotopic.

We prove by induction on $n \geq 0$. Since P_0 is projective, the following commutative diagram proves the existence of f_0

$$\begin{tikzcd}
& P_0 \arrow[dl, dashed, "f_0"'] \arrow[d, "f\partial_0"] \\
P'_0 \arrow[r, "\partial'_0"'] & A'
\end{tikzcd}$$

For the induction step, consider

$$\begin{tikzcd}
P_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, dashed] & P_n \arrow[r, "\partial_n"] \arrow[d, "f_n"] & P_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \dots \\
P'_{n+1} \arrow[r, "\partial'_{n+1}"] & P'_n \arrow[r, "\partial'_n"] & P'_{n-1} \arrow[r] & \dots
\end{tikzcd}$$

We can have this implication due to projectivity of P_{n+1}

$$\begin{tikzcd}
 & P_{n+1} \arrow[dl, dashed, "f_{n+1}"'] \arrow[d, "f_n\partial_{n+1}"] & \\
P'_{n+1} \arrow[r, "\partial'_{n+1}"'] & \operatorname{im} \partial'_{n+1} \arrow[r] & 0
\end{tikzcd}$$

only if $\operatorname{im}\left(f_n\partial_{n+1}\right) \subset \operatorname{im} \partial'_{n-1} = \ker \partial'_n$, which is true since $\partial'_n f_n \partial_{n+1} = f_{n-1}\partial_n\partial_{n+1} = f_{n-1}0 = 0$. Thus by induction we have proved the existence of the chain map $f_\#$. On to its uniqueness up to homotopy equivalence.

Consider the chain complexes where we have to construct the homotopy map $s_\#$,

$$\begin{tikzcd}
{} & P_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[dl, dashed, "s_{n+1}"'] \arrow[d, shift right, "h_{n+1}"'] \arrow[d, shift left, "f_{n+1}"] & P_n \arrow[r, "\partial_n"] \arrow[dl, "s_n"'] \arrow[d, shift right, "h_n"'] \arrow[d, shift left, "f_n"] & P_{n-1} \arrow[r] \arrow[dl, "s_{n-1}"'] \arrow[d, shift right, "h_{n-1}"'] \arrow[d, shift left, "f_{n-1}"] & \dots \arrow[r, "\partial_3"] & P_2 \arrow[r, "\partial_2"] \arrow[dl, "s_2"'] \arrow[d, shift right, "h_2"'] \arrow[d, shift left, "f_2"] & P_1 \arrow[r, "\partial_1"] \arrow[dl, "s_1"'] \arrow[d, shift right, "h_1"'] \arrow[d, shift left, "f_1"] & P_0 \arrow[r, "\partial_0"] \arrow[dl, "s_0"'] \arrow[d, shift right, "h_0"'] \arrow[d, shift left, "f_0"] & A \arrow[r, "\eta"] \arrow[dl, "s_{-1}"'] \arrow[d, "f"] & 0 \arrow[dl, "s_{-2}"'] \\
{} & P'_{n+1} \arrow[r, "\partial'_{n+1}"] & P'_n \arrow[r, "\partial'_n"] & P'_{n-1} \arrow[r] & \dots \arrow[r, "\partial_3"] & P'_2 \arrow[r, "\partial'_2"] & P'_1 \arrow[r, "\partial'_1"] & P_0 \arrow[r, "\partial'_0"] & A' \arrow[r, "\eta'"] & 0
\end{tikzcd}$$

such that $s_{n-1}\partial_n + \partial_{n+1}s_n = h_n - f_n \; \forall \; n \in \mathbb{N} \cup \{-2,-1,0\}$. Putting $s_{-1} = 0$ gives this for $n = 0$. As for the inductive step, we have to show that the s_n's so constructed satisfies $h_{n+1} - f_{n+1} - s_n\partial_{n+1} = \partial'_{n+2}s_{n+1}$ for the $(n+1)$th level. We have that $\left(h_{n+1} - f_{n+1} - s_n\partial_{n+1}\right)\left(P_{n+1}\right) \subset P'_{n+1}$. Only if we can show that $\left(h_{n+1} - f_{n+1} - s_n\partial_{n+1}\right)\left(P_{n+1}\right) \subset \operatorname{im} \partial'_{n+2} = \ker \partial'_{n+1}$, then by surjectivity of ∂'_{n+2} on its codomain, we have the following commutative diagram from where the projectivity of P_{n+1} implies the existence of s_{n+1} with the desired property.

$$\begin{tikzcd}
 & P_{n+1} \arrow[dl, dashed, "s_{n+1}"'] \arrow[d, "h_{n+1}-f_{n+1}-s_n\partial_{n+1}"] & \\
P'_{n+2} \arrow[r, "\partial_{n+2}"] & P'_{n+1} \arrow[r] & 0
\end{tikzcd}$$

$$\begin{aligned}
\text{But } \partial'_{n+1}\left(h_{n+1} - f_{n+1} - s_n\partial_{n+1}\right) &= \partial'_{n+1}\left(h_{n+1} - f_{n+1}\right) - \left(\partial'_{n+1}s_n\right)\partial_{n+1} \\
&= \partial'_{n+1}\left(h_{n+1} - f_{n+1}\right) - \left(h_n - f_n - s_{n-1}\partial_n\right)\partial_{n+1} \quad [\text{by induction hypothesis}] \\
&= 0, \text{ since } h \text{ and } f \text{ are chain maps.} \square
\end{aligned}$$

This being established, we try to see where this homotopy maps through the Hom functor. Of course, for $\varphi : P'_n \to G$, we get a map $\varphi \circ h_n : P_n \to G$, so h_n induces $\theta : \operatorname{Hom}\left(P'_n, G\right) \to \operatorname{Hom}\left(P_n, G\right); \varphi \mapsto \varphi \circ h_n$. In a similar manner all the chain and homotopy maps are transferred to our commutative diagram through Hom

$$\begin{tikzcd}
\operatorname{Hom}\left(P_{n-1}, G\right) \arrow[r, "\Delta_n"] & \operatorname{Hom}\left(P_n, G\right) & \\
 & \operatorname{Hom}\left(P'_n, G\right) \arrow[ul, "\sigma_{n-1}"] \arrow[u, shift right, "\theta_n"'] \arrow[u, shift left, "\varphi_n"] \arrow[r, "\Delta'_{n+1}"] & \operatorname{Hom}\left(P'_{n+1}, G\right) \arrow[ul, "\sigma_n"']
\end{tikzcd}$$

In fact, a cochain complex is induced. As we know, Hom is a contravariant functor, and as shown below, the chain homotopy condition obtained previously gives $\theta_n - \varphi_n = \sigma_n\Delta'_{n+1} + \Delta_n\sigma_{n-1}$, which gives the cochain homotopy relation between the two cochains obtained through Hom functor.

Now, if $A = A'$, f is instead an isomorphism, and the bottom row is projective as well, i.e., we are dealing with two projective resolutions of A. An ismorphism f induces isomorphisms $P_n \to P'_n$ by inducing homomorphisms both ways. All induced chain maps have been seen to be homtopic to the isomorphism, i.e., resolutions $P_\#$ and $P'_\#$ are homotopically equivalent. We know that cohomology groups of homotopic cochain maps are isomorphic. So the Ext functors, which are cohomology groups of the Hom cochains of the resolutions, are the same. That is, they are independent of the particular resolution chosen over A, and instead just depend on A and G– but also on the ring R over which the homomorphisms are taken.