A BASIC INTRODUCTION TO FUZZY TOPOLOGY

Contents

Abstract. We explain the concept of fuzzy sets as introduced by L. A. Zadeh in 1965 [15]. We first discuss the backdrop of this concept and how the notion of fuzzy sets can be applied, then some set-theoritic definitions and results shall be extended to fuzzy sets. Further, algebraic operations and the underlying meaning of their application to fuzzy sets shall be discussed.
Then, we explore the extensions of concepts from general topology to fuzzy sets, and discuss sequences, continuity and compactness of sets in fuzzy topological spaces.
This was the report I submitted in the third semester of my M.Sc., at National Institute of Technology Silchar, while I was still working on my M.Sc. thesis under the supervision of Dr. Juthika Mahanta.

1. Introduction to Fuzzy sets

1.1. Backdrop.

> 'But,' you might say, 'none of this shakes my belief that 2 and 2 are 4.' You are quite right, except in marginal cases – and it is only in marginal cases that you are doubtful whether a certain animal is a dog or a certain length is less than a metre. Two must be two of something, and the proposition '2 and 2 are 4' is useless unless it can be applied. Two dogs and two dogs are certainly four dogs, but cases arise in which you are doubtful whether two of them are dogs. 'Well, at any rate there are four animals,' you might say. But there are microorganisms concerning which it is doubtful whether they are animals or plants. 'Well, then living organisms,' you say. But there are things of which it is doubtful whether they are living or not. You will be driven into say: 'Two entities and two entities are four entities.' When you have told me what you mean by 'entity', we will resume the argument.

Simon Singh, *Fermat's Last Theorem, The story of a riddle that confounded the world's greatest minds for 358 years*, Fourth Estate, London, 1997.

. While we try to deal mathematically with the problems in the world, we come across:

(1) Analytic methods based on calculus of dealing with physical problems are applicable only to problems involving a handful of variables related to one another in a predictable way. According to Warren Waver [1948], 'problems of organized simplicity'.
(2) Statistical methods require a large number of variables with a high degree of randomness. According to Weaver, problems of 'disorganized complexity.'
(3) 'Organized complexity' or Non-linear systems with large number of components and rich interactions among the components, which are usually non-deterministic, but not as a result of randomness that could yield meaningful statistical averages

Unbelievable accleration in evolution of technology during World War II, including Alan Turing's ground-breaking experiments with his idea of 'the universal computer', made scientists believe for a span of time, that the level of complexity we can handle is basically a matter of the level of computational power at our disposal.
. But, as Dr. Paul Erdös is seen to comment in the film *N is a Number* dir. George Paul Ciscery, 'You can say, how do we not know? We have these powerful computers! But, they are not good enough.' Or as Hans Bremermann proposed in 1962, 'No data processing system, whether artifical or living, can process more than 2×10^{47} bits per second per gram of its mass.' That leaves us with unsolved calculations if such impossibly fast computers were being run since the formation of the Earth.
. Traditionally, *uncertainty* has been tackled by *probabilty theory*. But in the 1960s, other approaches to this end began to be explored, among which L. A. Zadeh's paper is certainly worth mentioning. Some ideas presented in that paper were suggested in 1937 by American philosopher Max Black. In that paper, Zadeh broke away from Aristotelian two-valued logic and introduced a "fuzzy" set, where membership is not a matter of affirmation or denial, but rather a matter of degree.

Let us consider an ordinary cup, which is clearly a cup and not a plate. But what about a tablet? In some way it is a computer and in some other way it can simply work as a mobile phone. But then it is not as good as a laptop for my work, and it is too clumsy to carry for a phone. Therefore, absolute "belonging" to either set is not applicable here. This brings us to a new realm of the basic scientific instinct of classification.

Definition 1.1.1. *Let Ω be a space of points with a generic element that we denote by x.*
A fuzzy set (class) *$A \subseteq \Omega$ is characterized by a* membership (characteristic) function *$\mu_A : \Omega \to [0,1]$ with the value of $\mu_A(x)$ denoting the* grade of membership *of x in A.*

$$A = \{(x, \mu_A(x)) \mid x \in \Omega\} \tag{1}$$

. In our previous non-mathematical example, the tablet might have a value of membership function with respect to set of computers as 0.4, and that with respect to mobile phones as 0.65. An ipad pro might be more versatile like a computer, e.g., $\mu_{\text{set of computers}}$(ipad pro) might have value 0.6. But clearly for our cup, membership function has value 1 or 0, accordingly as the set is that of cups or not. This is called *ordinary* or *crisp set* whose membership function takes only 2 values.

What about probability?

- Traditionally, uncertainty has been tackled by probability theory.
- Both fuzziness and probability give values in the continuum $[0,1]$.
- But understanding what the two systems are trying to model, we see that they are providing us with two entirely different aspects of information.

An example might be appropriate here. Let $\mathcal{P}$ be the set of all potable liquids.

$\mu_{\mathcal{P}}(K) = 0.91$

FIGURE 1. Bottle K

$P(M \in \mathcal{P}) = 0.91$

FIGURE 2. Bottle M

Which one will you choose to drink from? The answer, obviously, is K, if one wants to avoid the risk of drinking something like hydrochloric acid, probability of which being in M is 9%. But K, which contains something more or less like drinking water, can be filled with at most swamp water or coke.

1.2. **Set terminology.**

1.2.1. *Set Operations.*

Definition 1.2.1. (1) *A is an* empty *fuzzy set in Ω iff $\mu_A(x) = 0 \ \forall \ x \in \Omega$.*
(2) *Two fuzzy sets A and B are* equal, *i.e., $A = B$ iff $\mu_A(x) = \mu_B(x) \ \forall \ x \in \Omega$.*
(3) Complement *of a fuzzy set A is denoted by A' and is defined by*

$$\begin{aligned} \mu_{A'}(x) &= 1 - \mu_A(x) \ \forall \ x \in \Omega \\ \textit{or, } \mu_{A'} &= 1 - \mu_A. \end{aligned}$$

(4) ***Containment.*** *A fuzzy set A is* contained *in a fuzzy set B or A is a* subset of B *or A is* smaller than or equal to B *iff*

$$\mu_A \leq \mu_B.$$

Definition 1.2.2. *Let A and B be two fuzzy sets with membership functions $\mu_A(x)$ anf $\mu_B(x)$ respectively. Their* union *is a fuzzy set $C = A \cup B$ whose membership function μ_C is given by*

$$\begin{aligned} \mu_C(x) &= \max\{\mu_A(x), \mu_B(x)\} \ \ \forall \ x \in \Omega, \\ \textit{or, } \mu_C &= \mu_A \vee \mu_B. \end{aligned} \tag{2}$$

Associative property.

$$\begin{aligned} \mu_{A\cup(B\cup C)}(x) &= \max\{\mu_A(x), \max\{\mu_B(x), \mu_C(x)\}\} \ \ \forall \ x \in \Omega \\ &= \max\{\max\{\mu_A(x), \mu_B(x)\}, \mu_C(x)\} \ \ \forall \ x \in \Omega \\ &= \mu_{(A\cup B)\cup C}(x) \ \forall \ x \in \Omega, \\ \text{i.e., } A \cup (B \cup C) &= (A \cup B) \cup C. \end{aligned}$$

Definition 1.2.3 (Alternative definition)**.** *The* union *of fuzzy sets A and B is the smallest fuzzy set containing both A and B.*

Two definitions are equivalent. From (1), since

$$\mu_C(x) = \max\{\mu_A(x), \mu_B(x)\} \geq \mu_A(x) \ \forall \ x \in \Omega$$

and

$$\mu_C(x) = \max\{\mu_A(x), \mu_B(x)\} \geq \mu_B(x) \ \forall \ x \in \Omega,$$

$$A, B \subseteq C.$$

This is true the other way as well. Furthermore, if D be another fuzzy set $\ni A, B \subseteq D$ then $\mu_D \geq \mu_A, \mu_B$, hence

$$\mu_D \geq \max\{\mu_A, \mu_B\} = \mu_C \Longleftrightarrow C \subseteq D.$$

Definition 1.2.4. *Let A and B be fuzzy sets with respective membership functions $\mu_A(x)$ and $\mu_B(x)$ respectively. Then their intersection $C = A \cap B$ has membership function*

$$\mu_C = \min\{\mu_A, \mu_B\} \ \ \textit{or, } \mu_C = \mu_A \wedge \mu_B.$$

Alternatively, the intersection $A \cap B$ is the largest fuzzy set which is contained in both A and B.

Definition 1.2.5. *Two fuzzy sets A and B are* disjoint *if $A \cap B$ is empty.*

Union and intersection graphically:

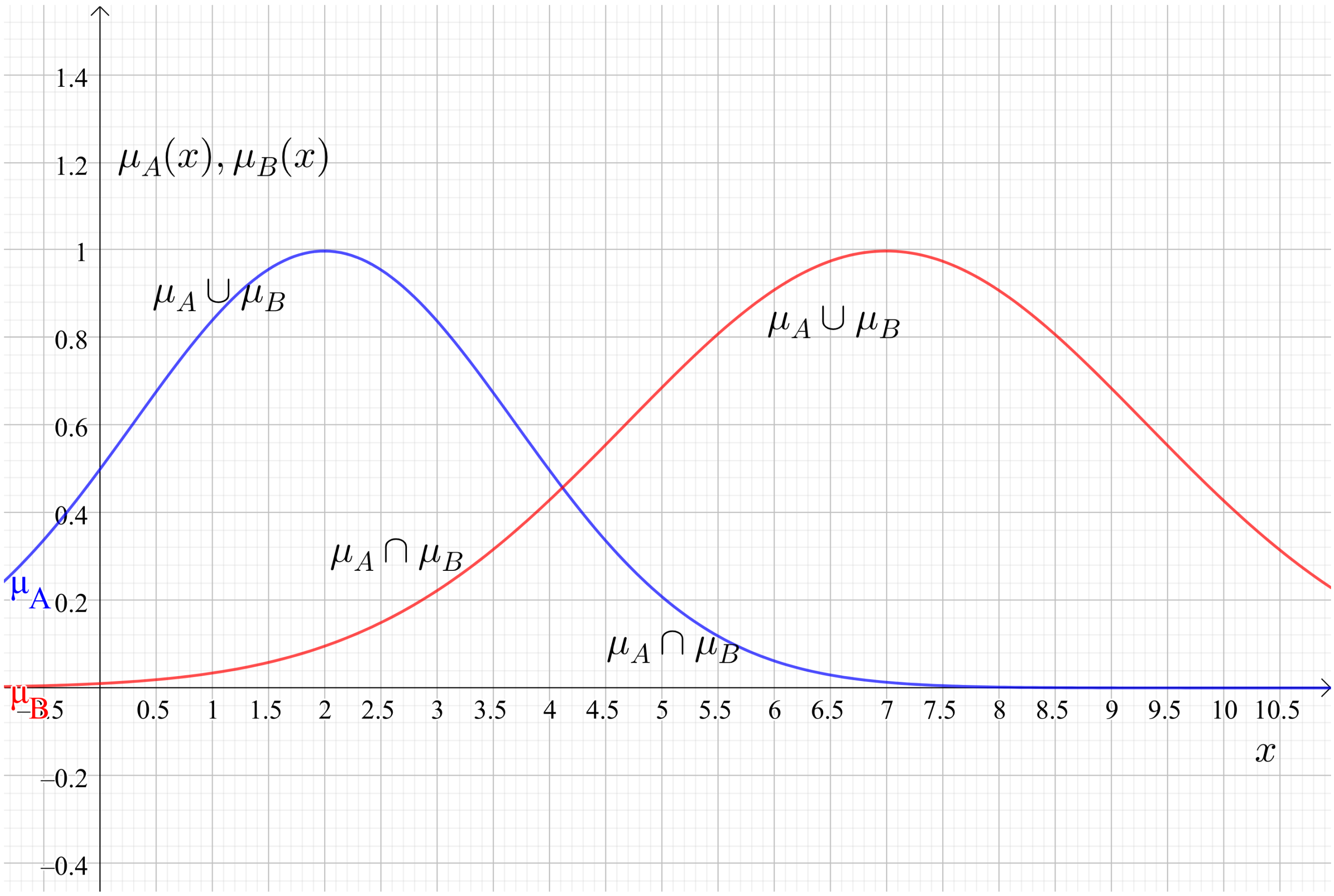


FIGURE 3. ▲ Union and intersection of fuzzy sets in $\mathbb{R}^1$

In general, for a family of fuzzy sets $A = \{A_i \mid i \in I\}$, the union $C = \cup_{i\in I} A_i$ and the intersection $D = \cap_{i\in I} A_i$ are defined by

$$\begin{aligned} \mu_C(x) &= \sup_{i\in I} \{\mu_{A_i}(x)\}, \ x \in X; \\ \mu_C(x) &= \inf_{i\in I} \{\mu_{A_i}(x)\}, \ x \in X. \end{aligned}$$

We know that, ordinary set $C = \varphi(A_1, A_2, \ldots, A_n)$ (where φ is a function made of connectives of the form $\cap$ or $\cup$) can be represented as a network of switches α_i corresponding to A_i where

(1) $A_i \cap A_j$ represents series combination of α_i and α_j,
(2) $A_i \cup A_j$ represents parallel combination of α_i and α_j.

For fuzzy sets, let $\mu_i(x); i = 1, \ldots, n$ denote the membership function of A_i at x. We associate with $\mu_i(x)$ a sieve $S_i(x)$ whose mesh is of size $\mu_i(x)$. Then

(1) $A_i \cap A_j$ or $\mu_i(x) \wedge \mu_j(x)$ represents series combination of $S_i(x)$ and $S_j(x)$,
(2) $A_i \cup A_j$ or $\mu_i(x) \vee \mu_j(x)$ represents parallel combination of $S_i(x)$ and $S_j(x)$.

$$\boxed{C = ((A_1 \cup A_2) \cap A_3) \cup (A_4 \cap A_5)}$$

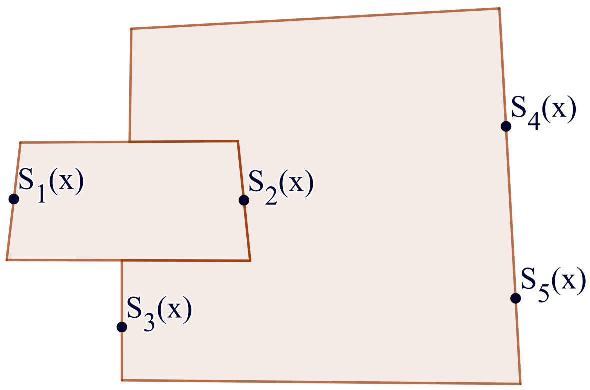


FIGURE 4. A network of sieves simulating $((\mu_1(x) \vee \mu_2(x)) \wedge \mu_3(x)) \vee (\mu_4(x) \wedge \mu_5(x))$

1.2.2. *Laws of set theory.*
. In one way, we can say that a point x "belongs" to a fuzzy set A when $\mu_A(x) > 0$. Less trivially, we can introduce 2 levels $0 < \beta < \alpha < 1$ such that

(1) $x \in A$ if $\mu_A(x) \geq \alpha$,
(2) $x \notin A$ if $\mu_A(x) \leq \beta$,
(3) x has indeterminate status relative to A if $\beta < \mu_A(x) < \alpha$.

This leads to a three-valued logic as in [8] with 3 truth values:

$$T\left(\mu_A(x) \geq \alpha\right), F\left(\mu_A(x) \leq \beta\right), U\left(\beta < \mu_A(x) < \alpha\right).$$

.

Theorem 1.2.6 (De Morgan's laws)**.** *If A and B are fuzzy sets, with notation used previously,*

$$(A \cup B)' = A' \cap B', \tag{3}$$
$$(A \cap B)' = A' \cup B'. \tag{4}$$

These are equivalent to the statements:

$$\begin{aligned} 1 - \max\{\mu_A, \mu_B\} &= \min\{1 - \mu_A, 1 - \mu_B\}, \\ 1 - \min\{\mu_A, \mu_B\} &= \max\{1 - \mu_A, 1 - \mu_B\}. \end{aligned}$$

Similarly, we can say

$$\max\{\mu_C, \min\{\mu_A, \mu_B\}\} = \min\{\max\{\mu_C, \mu_A\}, \max\{\mu_C, \mu_B\}\}$$

and hence come up with

Theorem 1.2.7 (Distributive laws)**.**

$$C \cap (A \cup B) = (C \cap A) \cup (C \cap B), \tag{5}$$
$$C \cup (A \cap B) = (C \cup A) \cap (C \cup B). \tag{6}$$

Essentially, fuzzy sets in Ω constitute a distributive lattice with a 0 and 1. [1]

Observation 1.2.8. [13] *For crisp set A,*
Law of Contradiction: $A \cap A' = \phi$
Law of Excluded Middle: $A \cup A' = \Omega$.
But for a non-crisp fuzzy set $\tilde{A} = (A, \mu_A)$,

$$\begin{aligned} \mu_{A\cap A'} &= \min\{\mu_A, 1-\mu_a\} \neq 0, \\ \mu_{A\cup A'} &= \max\{\mu_A, 1-\mu_a\} \neq 1. \end{aligned}$$

$\therefore$ *these laws do not necessarily hold for fuzzy sets.*

1.3. Algebraic operations.

Definition 1.3.1 (Algebraic product)**.** *The* algebraic product *of A and B is denoted by AB and is defined in terms of membership functions as*

$$\mu_{AB} = \mu_A \mu_B. \tag{7}$$

Since $0 \leq \mu_A, \mu_B \leq 1 \implies \mu_{A\cap B} = \min\{\mu_A, \mu_B\} \geq \mu_A\mu_B$, (" $=$ " iff $\mu_A, \mu_B \in \{0, 1\}$) we can say,

Result 1.3.2.

$$AB \subseteq A \cap B. \tag{8}$$

The dual of AB is the *sum* $A \oplus B = (A'B')' = A + B - AB$. For ordinary sets, $\cap \approx$ algebraic product, and $\cup \approx \oplus$.

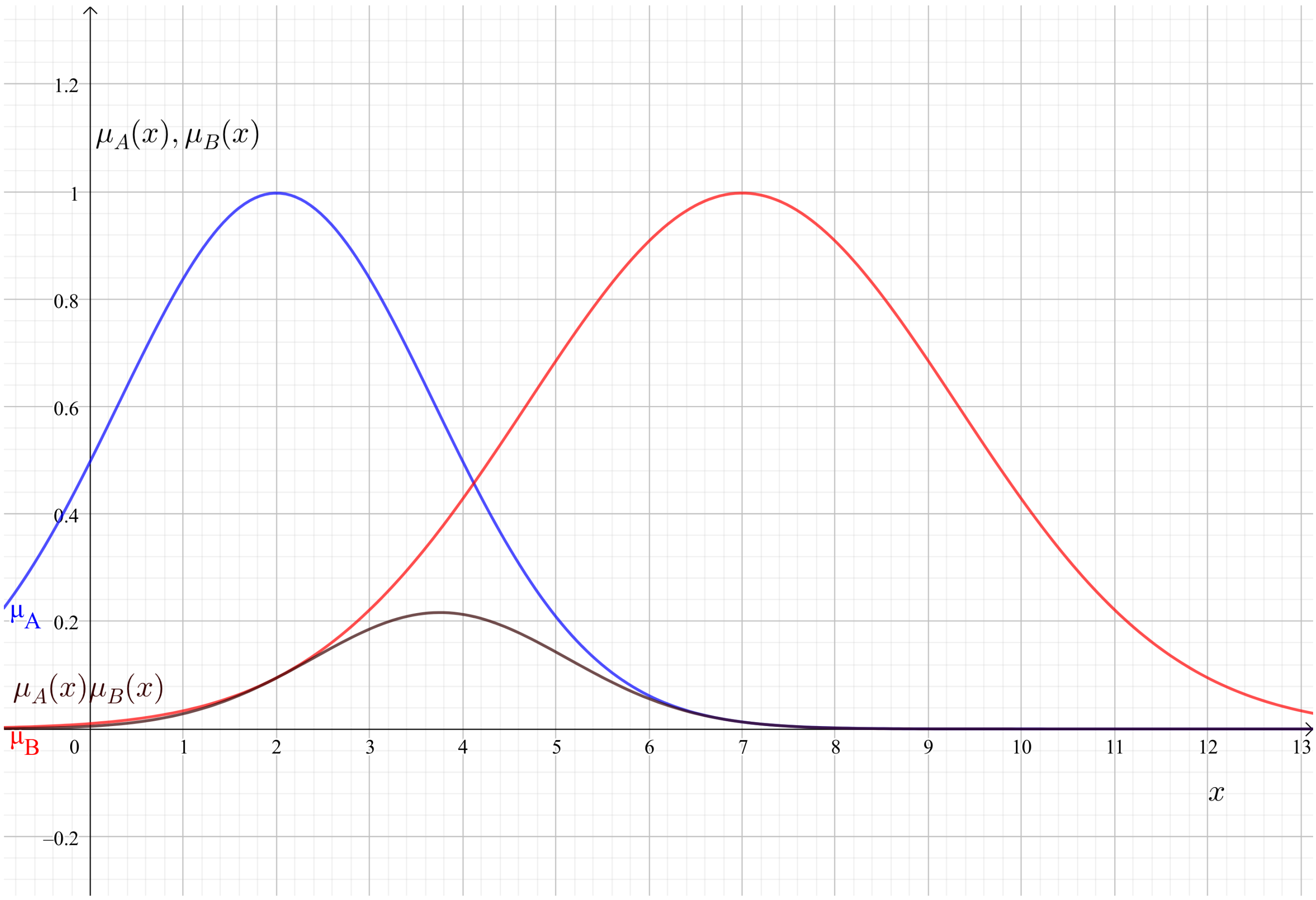


Figure 5. Product of two fuzzy sets A and B in $\mathbb{R}^1$

Definition 1.3.3 (Algebraic sum)**.** *The algebraic sum $A + B$ of A and B is defined by*

$$\mu_{A+B} = \mu_A + \mu_B \tag{9}$$

provided $\mu_A(x) + \mu_B(x) \leq 1 \; \forall \; x$.

Since this definition does not make much sense if $\mu_A(x) + \mu_B(x) \geq 1$, we can otherwise define algebraic sum as in [16], that

Definition 1.3.4. *The algebraic (probabilistic) sum* $\tilde{C} = \tilde{A} \oplus \tilde{B} = \{(x, \mu_{A\oplus B}(x)) \mid x \in \Omega\}$ *is defined by membership function*

$$\mu_{A\oplus B} = \mu_A + \mu_B - \mu_A\mu_B \leq 1. \tag{10}$$

Or else,

Definition 1.3.5 (Bounded sum)**.** *The* bounded sum *of* A *and* B *is defined as*

$$\tilde{C} = \tilde{A} + \tilde{B} = \{(x, \mu_{A+B}(x)) \mid x \in \Omega\}$$

where

$$\mu_{A+B} = \min\{1, \mu_A + \mu_B\}. \tag{11}$$

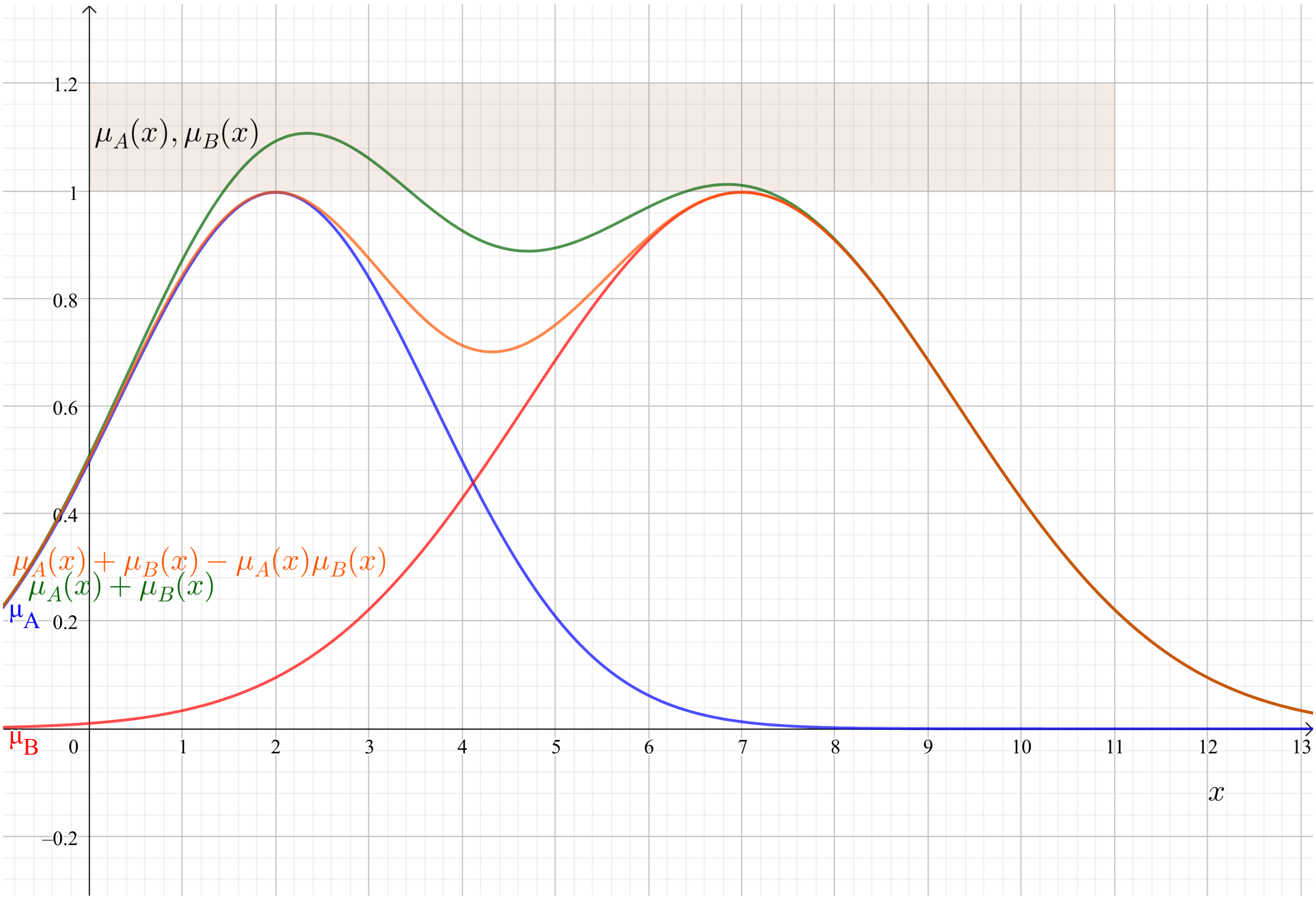


FIGURE 6. Comparison of sums of two fuzzy sets A and B in $\mathbb{R}^1$

Definition 1.3.6 (Absolute difference)**.** *The* absolute difference $|A - B|$ *of* A *and* B *is defined by*

$$\mu_{|A-B|} = |\mu_A - \mu_B|. \tag{12}$$

For ordinary sets,

$$\begin{aligned} x \notin |A - B| &\iff \mu_{|A-B|}(x) = 0 \\ &\iff x \in A, B \text{ or } x \notin A, B \\ &\iff x \notin (A \cup B) \setminus (A \cap B) \end{aligned}$$

and

$$\begin{aligned} x \in |A-B| &\iff \mu_{|A-B|}(x) = 1 \\ &\iff x \in A, x \notin B \text{ or } x \notin A, x \in B \\ &\iff x \in (A \cup B) \setminus (A \cap B). \end{aligned}$$

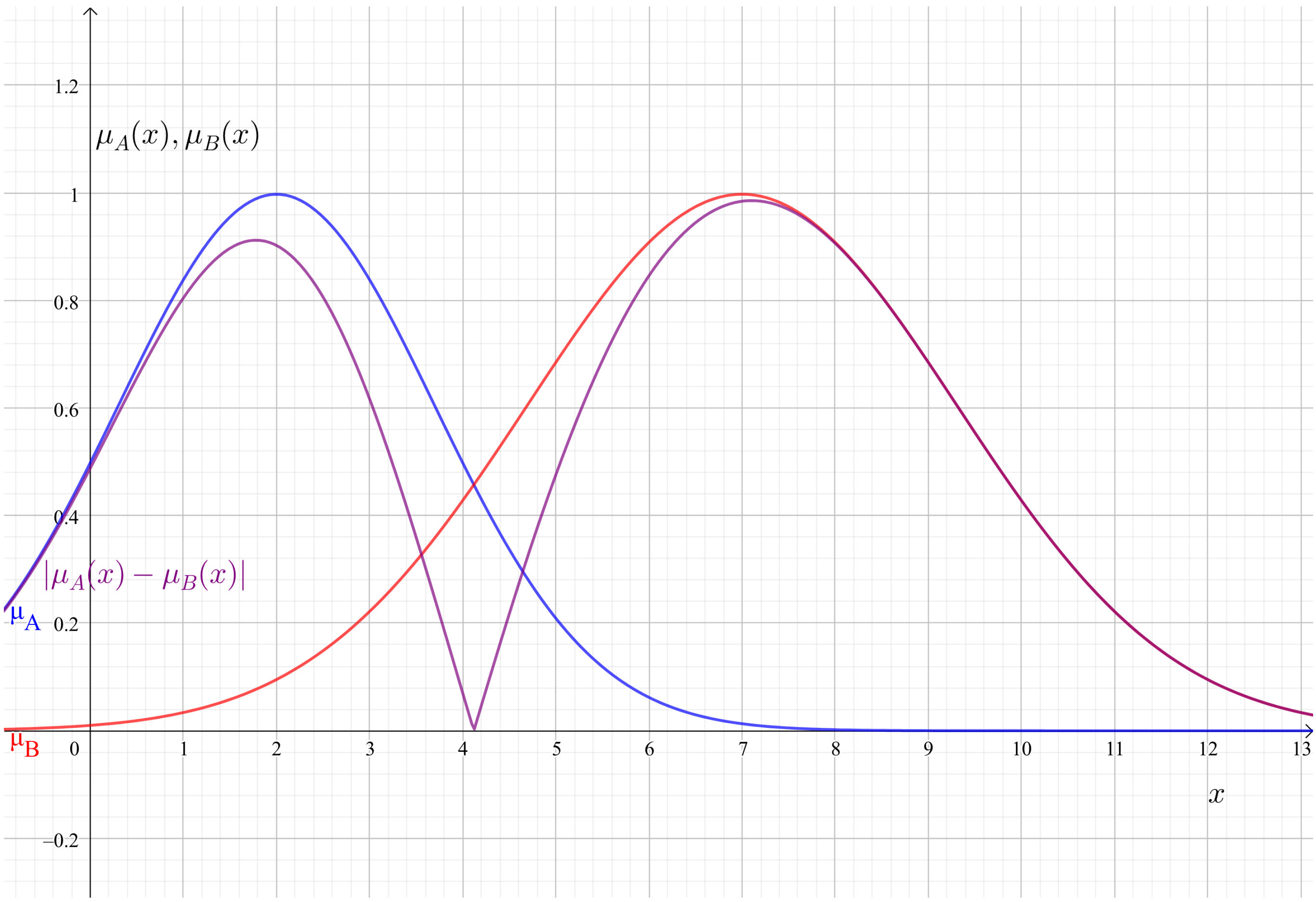


FIGURE 7. Absolute difference of two fuzzy sets A and B in $\mathbb{R}^1$

Definition 1.3.7 (Convex combination)**.** *The* convex combination of arbitrary fuzzy sets A, B, Λ *is denoted by* $(A, B; \Lambda)$ *and is defined by the relation:*

(13) $$(A, B; \Lambda) = \Lambda A + \Lambda' B.$$

In terms of membership functions,

(14) $$\mu_{(A,B;\Lambda)}(x) = \mu_\Lambda(x)\mu_A(x) + [1 - \mu_\Lambda(x)]\mu_B(x); x \in \Omega.$$

Result 1.3.8. *For fuzzy sets A, B,*

(15) $$A \cap B \subset (A, B; \Lambda) \subset A \cup B \ \forall \Lambda$$

Proof. If $\lambda \in [0, 1]$,

$$\begin{aligned} \lambda\mu_A(x) + (1-\lambda)\mu_B(x) &\geq \min\{\mu_A(x), \mu_B(x)\} \\ \lambda\mu_A(x) + (1-\lambda)\mu_B(x) &\leq \max\{\mu_A(x), \mu_B(x)\}, \ x \in \Omega. \end{aligned}$$

Now, we have another interesting observation:

Proposition 1.3.9. *Given any fuzzy set $C \ni A \cap B \subseteq C \subseteq A \cup B$, $\exists\ \Lambda \ni C = (A, B; \Lambda)$.*

Indeed, the membership function of Λ can be computed as:

(16) $$\mu_\Lambda(x) = \frac{\mu_C(x) - \mu_B(x)}{\mu_A(x) - \mu_B(x)}, \ x \in \Omega.$$

1.4. **Fuzzy sets induced by mappings.**

Definition 1.4.1 (Relations)**.** *A* relation *is defined as a set of ordered pairs.* [4]

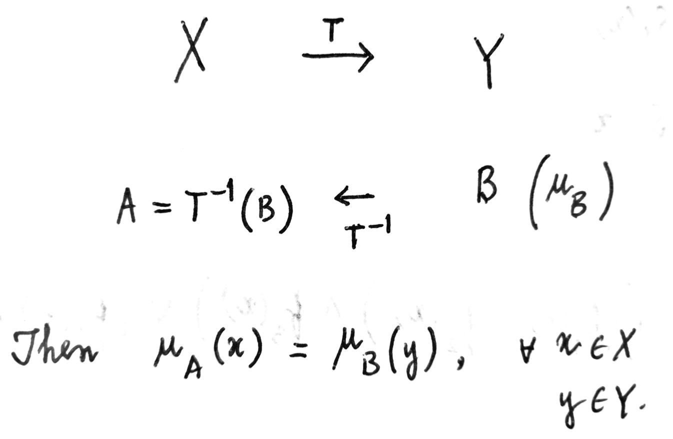
Definition 1.4.2. *A* fuzzy relation in Ω *is a fuzzy set in the product space* $\Omega \times \Omega$*. An* n-ary fuzzy relation *in* Ω *as a fuzzy set* A *in the product space* $\Omega \times \Omega \times \cdots \times \Omega$ *(*n *times).*

Definition 1.4.3 (Composition)**.** *Composition of two fuzzy relations* A *and* B *is denoted by* $B \circ A$*, which is a fuzzy relation in* Ω *defined in terms of membership functions as*

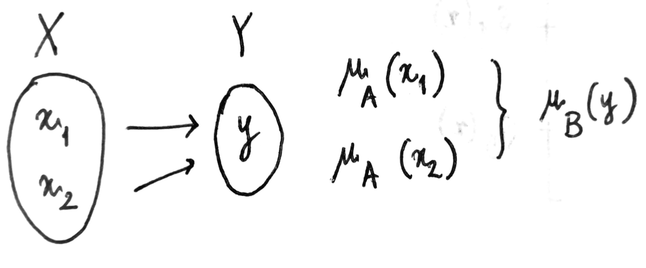
$$\mu_{B \circ A}(x, y) = \sup_{v} \min \{\mu_A(x, v), \mu_B(v, y)\}, v \in \Omega. \tag{17}$$

Note. $A \circ (B \circ C) = (A \circ B) \circ C$.

Let T be a mapping from space X to space Y, and let us consider fuzzy set $B \subseteq Y$ and $A = T^{-1}(B) \subseteq X$. What will be the membership function of a fuzzy set $B \in Y$ induced by mapping T from a fuzzy set $A \in X$?



Ambiguity shall arise when T is not an injective map, i.e., when we shall have $x_1, x_2 \in X \ni T(x_1) = T(x_2) = y \in Y$. Then what will be the value of $\mu_B(y)$? $\mu_A(x_1)$ or $\mu_A(x_2)$?



The membership function for B is defined by:

$$\mu_B(y) = \begin{cases} \max\limits_{x \in T^{-1}(y)} \mu_A(x), y \in Y & \text{if } T^{-1}\{y\} \neq \phi, \\ 0 & \text{if } T^{-1}\{y\} = \phi. \end{cases} \tag{18}$$

Theorem 1.4.4 (Properties of fuzzy sets induced by mappings)**.** [2, Theorem 4.1] *Let* $f : X \to Y$ *and* $g : Y \to Z$ *be two mappings where* X*,* Y *and* Z *are spaces. Let* $g \circ f$ *be the composition of* f *and* g*. Let* $A_1 \subseteq X, A_2 \subseteq X, B_1 \subseteq Y, B_2 \subseteq Y, C \subseteq Z$ *be arbitrary fuzzy sets. Then,*

(1) $f^{-1}[B_1'] = (f^{-1}[B_1])'$.
(2) $f[A_1'] \supseteq (f[A_1])'$.
(3) $B_1 \subseteq B_2 \implies f^{-1}[B_1] \subseteq f^{-1}[B_2]$.
(4) $A_1 \subseteq A_2 \implies f[A_1] \subseteq f[A_2]$.
(5) $B_1 \supseteq f[f^{-1}[B_1]]$.
(6) $A_1 \subseteq f^{-1}[f[A_1]]$.
(7) $(g \circ f)^{-1}[C] = f^{-1}[g^{-1}[C]]$.

1.5. **Partition, Convexity, Shadow and some results.** We assume from now on that Ω is a real Euclidean space E^n.

Definition 1.5.1 (Partition of S for grade α)**.** *The crisp set containing only elements belonging with a grade of membership of at least α to a fuzzy set in question, i.e.,*

$$\Gamma_\alpha(S) = \{x \in \Omega \mid \mu_S(x) \geq \alpha\} \tag{19}$$

This is also called the weak α-cut *of S. The* strong α-cut *of S is defined as*

$$\sigma_\alpha(S) = \{x \in \Omega \mid \mu_S(x) > \alpha\}\,. \tag{20}$$

$\sigma_0(S)$ is called the support *of A. It is the crisp set of all elements of Ω which have non-zero membership functions with respect to A.*

Definition 1.5.2 (Convexity)**.** (1) *A fuzzy set S in a linear psace Ω is said to be* convex *if the crisp set Γ_α are convex $\forall\ \alpha \in [0,1]$.*

Basically here, we define S to be convex iff for any $x, y \in S$, all elements that can be expressed as a convex combination of x and y have at least as high a grade of membership to S as either x or y.

(2) *A fuzzy set S is said to be convex if $\forall\ \lambda \in (0,1)$*

$$\mu_S\left(\lambda x + (1-\lambda)y\right) \geq \min\left\{\mu_S(x), \mu_S(y)\right\}. \tag{21}$$

Result 1.5.3 (Equivalence of the two definitions)**.**

Definition (1) $\implies$ Definition (2). Let us consider

$$\alpha = \min\left\{\mu(x), \mu(y)\right\} \tag{22}$$

where μ refers to μ_S. Now since Γ_α is convex, we have that for any $\lambda \in [0,1]$,

$$\lambda x + (1-\lambda)y \in \Gamma_\alpha$$

and for all $\gamma \in \Gamma_\alpha$ we have $\mu(\gamma) \geq \alpha = \min\left\{\mu(x), \mu(y)\right\}$. □

Definition (2) $\implies$ Definition (1). For any α we have $x, y \in \Gamma_\alpha \implies \mu\left(\lambda x + (1-\lambda)y\right) \geq \min\left\{\mu(x), \mu(y)\right\} = \alpha \implies \lambda x + (1-\lambda)y \in \Gamma_\alpha$, which implies convexity of the crisp sets Γ_α. □

Theorem 1.5.4. *If A and B are convex fuzzy sets then $A \cap B$ is convex.*

Proof. Let $C = A \cap B$. Then,

$$\begin{aligned}
\mu_C\left(\lambda x + (1-\lambda)y\right) &= \min\left\{\mu_A\left(\lambda x + (1-\lambda)y\right), \mu_B\left(\lambda x + (1-\lambda)y\right)\right\} \\
&\geq \min\left\{\min\left\{\mu_A(x), \mu_A(y)\right\}, \min\left\{\mu_B(x), \mu_B(y)\right\}\right\} \\
&= \min\left\{\min\left\{\mu_A(x), \mu_B(x)\right\}, \min\left\{\mu_A(y), \mu_B(y)\right\}\right\} \\
&= \min\left\{\mu_C(x), \mu_C(y)\right\}. \quad \square
\end{aligned}$$

Definition 1.5.5 (Strict convexity)**.** *A fuzzy set S is said to be* strictly convex *if for any $\alpha \in (0,1]$,*

$$\frac{1}{2}x + \frac{1}{2}y \in \Gamma_\alpha^{int}\ \forall\ x, y \in \Gamma_\alpha.$$

Definition 1.5.6 (Strong convexity)**.** *A fuzzy set S is said to be* strongly convex *if for any $\lambda \in (0,1)$ and for any distinct x and y,*

$$\mu_S\left(\lambda x + (1-\lambda)y\right) > \min\left\{\mu_S(x), \mu_S(y)\right\}.$$

. **Note.** Strict convexity does not imply strong convexity. The crisp set containing the unit disc in $\mathbb{R}^2$ provides an example of a strictly but not strongly convex fuzzy set.
Also, strong convexity does not imply strict convexity. But it does in $\mathbb{R}^1$. $x, y \in \Gamma_\alpha \implies (x, y) \subseteq \Gamma_\alpha$ by strong convexity and x, y form the boundary of that interval.

Theorem 1.5.7. *If A and B are strongly convex fuzzy sets in $\mathbb{R}^n$, then $C = A \cap B$ is strongly convex as well.*

The proof of this theorem is quite the same as that of the preservation of weaker convexity in case of intersection. Only the inequalities involved will be strict in this case.

Theorem 1.5.8 (a theorem which is less trivial:)**.** *If A and B are strictly convex fuzzy sets in $\mathbb{R}^n$, then $C = A \cap B$ is strictly convex as well.*

Proof. Let x and y be arbitrary points belonging to partition for grade α for both A and B, and z be their midpoint $\frac{1}{2}x + \frac{1}{2}y$. Then by strict convexity of both sets for every point u near z, $\mu_A(u) \geq \alpha, \mu_B(u) \geq \alpha$ and thus $\mu_C(u) = \min\{\mu_A(u), \mu_B(u)\} \geq \alpha$. Thus $u \in \Gamma_\alpha^{(C)}$, the partition by grade α of C, and since this is true $\forall$ u near z, $z \in \left(\Gamma_\alpha^{(C)}\right)^{\text{int}}$. □

Result 1.5.9. *The characteristic function of a strongly convex set has no zeroes.*

Proof. Let $\mu(z) = \mu\left(\lambda x + (1-\lambda)y\right) = 0 > \mu(x), \mu(y)$, a contradiction! □

Result 1.5.10. *The characteristic function of a strongly convex fuzzy set attains its supremum at no more than one point.*

Proof. We simply suppose that it attains its supremum at two points, join them with a line and see that strong convexity is violated for any point on that line. □

Definition 1.5.11 (Boundedness)**.** *A fuzzy set A is bounded iff the sets $\Gamma_\alpha = \{x \mid \mu_A(x) \geq \alpha\}$ are bounded in norm $\forall$ $\alpha > 0$, i.e., $\forall \alpha > 0$ $\exists$ some finite $R(\alpha) \ni \|x\| \leq R(\alpha)$ $\forall$ $x \in \Gamma_\alpha$.*

Definition 1.5.12 (Maximal grade)**.** *For a fuzzy set S, the maximal grade M is defined by*

$$M = \sup_{x \in \Omega} \mu_S(x). \tag{23}$$

Lemma 1.5.13. *Let A be a bounded fuzzy set and its maximal grade be M. Then $\exists$ $x_0 \in \Omega$ at which M is* essentially attained*.*

Definition 1.5.14 (Essential attainment)**.** *$c \in \Omega$ is said to be* essentially attained *at x_0 at c, if for every $\varepsilon > 0$, every neighbourhood of x_0 contains points in the set*

$$Q(\varepsilon) = \{x \mid \mu_A(x) \geq c - \varepsilon\},$$

i.e., every neighbourhood of x_0 contains some $x_i \ni \mu_S(x_i) \geq c - \varepsilon$.

Back to the lemma, however, M need not be "essentially attained" if it is literally attained.

Proof of the lemma. [The case of a crisp set containing a single point demonstrates that M need not be essentially attained in this case.] Now we suppose that $\nexists$ any such point and consider the sequence of crisp sets $\left\{\Gamma_{\alpha(n)}\right\}$ where $\alpha(n) = M - \frac{M}{n+1}$. By definition of M, for every finite n, $\exists$ a point with membership grade $> \alpha(n)$. Therefore, $\Gamma_{\alpha(n)} \neq \phi$ $\forall$ $n \in \mathbb{N}$. Since A is bounded, $\Gamma_{\alpha(1)}$ is bounded, i.e., $\overline{\Gamma_{\alpha(1)}}$ is compact. Now, $\alpha(n) \uparrow$ since $-\frac{M}{n+1} > -\frac{M}{n}$ and $\mu_A(x) \geq M - \frac{M}{n+1} \implies \mu_A(x) \geq M - \frac{M}{n} \implies \Gamma_{\alpha(n)} \subseteq \Gamma_{\alpha(n-1)}$. Therefore, $\Gamma_{\alpha(n)} \subseteq \Gamma_{\alpha(n-1)} \subseteq \overline{\Gamma_{\alpha(n-1)}} \subseteq \overline{\Gamma_{\alpha(1)}}$.

We consider the sequence $\{x_n\}$ where $x_n \in \Gamma_{\alpha(n)}$ is arbitrarily chosen. By Bolzano-Weierstrass' theorem, some subsequence of $\{x_n\}$ converges to some point $x_0 \in \overline{\Gamma_{\alpha(1)}}$.

Since no point x_n satisfies $\mu_S(x_n) = M$, for fixed x_k we can take N large enough that $x_{n>N} \neq x_k$ and thus every neighbourhood of x_0 must contain an infinite number of unique points in (x_n).

Since we can also take N high enough that every $x_{n>N}$ has a membership grade within ε distance of M, M is essentially attained at x_0. □

Definition 1.5.15 (Core of a fuzzy set)**.** *If S is a fuzzy set with maximum grade M, its* core *denoted by $C(S)$ is the crisp set consisting of all points at which M is essentially attained.*

Theorem 1.5.16. *The core of a convex fuzzy set is also convex.*

Proof. $M = \sup_{x \in \Omega} \mu_S(x)$,
$C(S) = \{x_0 \in S \mid \forall\ \varepsilon > 0\ \forall\ \delta > 0\ \exists\ x_i \in B_\delta(x_0) \ni \mu_S(x_i) \geq M - \varepsilon\}$.
$x, y \in S \implies \mu_S\left(\lambda x + (1-\lambda)y\right) \geq \min\{\mu_S(x), \mu_S(y)\}$.
$x_0, y_0 \in C(S) \subseteq S \implies \mu_S(\underbrace{\lambda x_0 + (1-\lambda)y_0}_{=z_0}) \geq \min\{\mu_S(x_0), \mu_S(y_0)\}$

Now, $\forall$ $\varepsilon > 0, \forall$ $\delta > 0$ $\exists$ $x_i \in B_\delta(x_0) \ni \mu_S(x_i) \geq M - \varepsilon$, $\forall$ $\varepsilon > 0, \forall$ $\delta > 0$ $\exists$ $y_i \in B_\delta(y_0) \ni \mu_S(y_i) \geq M - \varepsilon$.

$$\therefore \forall\ \varepsilon > 0, \forall\ \delta > 0\ \exists\ \delta_1, \delta_2 \ni \delta = \varphi(\delta_1, \delta_2)$$

$$\exists \left.\begin{array}{c} x_i \in B_{\delta_1}(x_0) \\ y_i \in B_{\delta_2}(y_0) \end{array}\right\} z_i \in B_\delta\left(\lambda x_0 + (1-\lambda)y_0\right)$$

$$\ni \mu_S(z_i) \geq \min\{\mu_S(x_i), \mu_S(y_i)\} \geq M - \varepsilon.$$

What is a hyperplane? Geometrically, a *hyperplane* is a subspace whose dimension is one less than that of its ambient space.

Lemma 1.5.17. *If A is a bounded set, then for each $\varepsilon > 0$ $\exists$ a hyperplane H such that $\mu_A(x) \le \varepsilon$ $\forall$ x on the side of H which does not contain the origin.*

Proof. We consider the set $\Gamma_\varepsilon = \{x \mid \mu_A(x) \ge \varepsilon\}$. By hypothesis, this set is contained in a sphere S of radius $R(\varepsilon)$. Let H be any hyperplane supporting S. Then, all points on the side of H which does not contain the origin lie outside or on S, and hence for all such points $\mu_A(x) \le \varepsilon$. □

Definition 1.5.18 (Shadow of a fuzzy set). *Firstly, $\Omega = E^n$. Let $A \subseteq E^n$, whose membership function is given by*

$$\mu_A(x) = \mu_A(x_1, x_2, \ldots, x_n) \tag{24}$$

We shall as of now define shadow *or projection of a fuzzy set only on a coordinate hyperplane,i.e., if the hyperplane H is normal to the i-th basis vector, i.e., a hyperplane of the form $H = \{x \mid x_i = 0\}$; $i \in \{1, \ldots, n\}$.*
Shadow *of A on H is defined as a fuzzy set $S_H(A) \subseteq E^{n-1}$ with membership function*

$$\mu_{S_H(A)}(x) = \mu_{S_H(A)}(x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) = \sup_{x_i \in E^1} \mu_A(x_1, x_2, \ldots, x_n). \tag{25}$$

We note as well that this definition is consistent with definition of mapping in (18).

2. Fuzzy Topology

[In this section I have included proofs of results only if they are alternative proofs to the ones in standard texts, or new proofs.]

2.1. Open sets.

Notation 2.1.1. *We shall denote by ϕ the* empty fuzzy set*, i.e., the set where the membership fucntion is identically equal to 0 for all $x \in X$. For the universal set X, we have, $\mu_X(x) = 1$ $\forall$ $x \in X$.*

Definition 2.1.2 (Fuzzy topology). *Let τ be a family of fuzzy sets in X such that*

(1) $\phi, X \in \tau$;
(2) $A, B \in \tau \implies A \cap B \in \tau$;
(3) $A_i \in \tau$ $\forall$ $i \in I \implies \cup_{i \in I} A_i \in \tau$.

Then τ is called a fuzzy topology *for X and (X, τ) is called a* fuzzy topological space *or* fts *for short.*

This essentially, as in our known definition for crisp sets, means that τ is a family of fuzzy sets in X such that *arbitrary unions* and *finite intersections* of sets in τ belong to τ. Condition (1) gets redundant.

Definition 2.1.3 (Open and closed sets). *A set A is τ-open if $A \in \tau$, while it is τ-closed if and only if A^C is τ-open.*

Notation 2.1.4. (1) Indiscrete fuzzy topology*, $I = \{\phi, X\}$;*
(2) Discrete fuzzy topology*, $D \supset \{\{x\} \mid x \in X\}$.*

Definition 2.1.5. *A topology υ on X is said to be* coarser *than a topology τ on X, or in other words, τ is said to be* finer *than υ, if $\upsilon \subset \tau$. Two topologies are* equal *if and only if both are finer, or coarser, than each other.*

. As of yet, we have been accustomed to consider neighbourhoods of points, i.e., a set $A \subseteq X$ is said to be a neighbourhood of a point $x \in X$ if $\exists$ $U^{\text{open}} \subseteq X \ni x \in U^{\text{open}} \subseteq A$. But when we come to think of fuzzy sets, since it is not a black-and-white fact that whether a point belongs to a set or not, it makes more sense to think of neighbourhoods of a fuzzy set itself (which contains points to the extent of the values of their membership functions).

Definition 2.1.6 (Neighbourhood of a fuzzy set). *A fuzzy set $U \subseteq X$ in an fts (X, τ) is said to be a* neighbourhood *(*nbhd *for short) of a fuzzy set $A \subseteq X$ if and only if there exists a τ-open fuzzy set O such that $A \subseteq O \subseteq U$.*
A neighbourhood system *of a fuzzy set is the family of all its neighbourhoods.*

Theorem 2.1.7. [2, Theorem 2.1] *A fuzzy set A is open if and only if A is a neighbourhood of each fuzzy set $B \subseteq A$.*

Theorem 2.1.8. [2, Theorem 2.2] *If υ is the nbhd system of a fuzzy set, then*

(1) $A_1, A_2, \ldots, A_n \in \upsilon \implies \cap_{i=1}^{n} A_i \in \upsilon \ (\infty > n \in \mathbb{N})$;
(2) *If a fuzzy set A be such that $A \supset B$ where $B \in \upsilon$, then $A \in \upsilon$.*

Definition 2.1.9. *Let A and B be fuzzy sets in an fts (X, τ) such that $A \supset B$. Then, B is an* interior fuzzy set *of A if and only if A is a nbhd of B.*
The interior *of A, $A^\circ = \{B \subseteq X \mid B$ is an interior fuzzy set of $A\}$.*

Theorem 2.1.10. [2, Theorem 2.3] *For a fuzzy set A, A° is the largest open fuzzy set contained in A. Moreover, A is open if and only if $A = A^\circ$.*

2.2. **Sequences of Fuzzy Sets.**

Definition 2.2.1. (1) *A sequence $\{A_n \mid n \in \mathbb{N}\}$ is said to be* eventually contained *in a fuzzy set A if and only if $\exists\ m \in \mathbb{N}\ \ni$*

$$n \geq m \implies A_n \subseteq A.$$

(2) *A sequence $\{A_n \mid n \in \mathbb{N}\}$ in an fts (X, τ) is said to* converge *to a fuzzy set $A \subseteq X$ if and only if the sequence is eventually contained in every nbhd of A. [Note here that the sequence* need not be *eventually contained in A.]*

Definition 2.2.2. (1) *A sequence $\{A_n \mid n \in \mathbb{N}\}$ is said to be* frequently contained *in A if and only if $\forall\ m \in \mathbb{N}\ \exists\ n \in \mathbb{N} \ni n \geq m$ and $A_n \subseteq A$.*
(2) *A fuzzy set A in a fts (X, τ) is said to be a* cluster fuzzy set *of a sequence of fuzzy sets if and only if that sequence is frequently contained in every nbhd of A.*

Definition 2.2.3 (Subsequence)**.** *A sequence $\{B_i \mid i \in \mathbb{N}\}$ is a subsequence of a sequence $\{A_n \mid n \in \mathbb{N}\}$ if and only if $\exists$ a mapping $N : \mathbb{Z} \cup \{0\} \to \mathbb{Z} \cup \{0\}\ \ni$*

(1) $B_i = A_{N(i)}$;
(2) $\forall\ m \in \mathbb{N},\ \exists\ n \in \mathbb{N}\ \ni$

$$i \geq n \implies N(i) \geq m.$$

Theorem 2.2.4. [2, Theorem 3.1] *If the nbhd system of each fuzzy set in an fts (X, τ) is countable, then*

(a) *Suppose we have a sequence $\{A_n \mid n \in \mathbb{N}\} \to B \subseteq A$. Then, $\{A_n \mid n \in \mathbb{N}\}$ is eventually contained in A if and only if A is open.*
(b) *If A is a cluster fuzzy set of a sequence $\{A_n \mid n \in \mathbb{N}\}$ of fuzzy sets, then $\exists$ a subsequence of $\{A_n \mid n \in \mathbb{N}\}$ that converges to A.*

Proof of part (b). Choose $m_1 \in \mathbb{N}$. $\exists\ n_1 \geq m_1 \ni A_{n_1} \subseteq A$. Choose $m_2 = n_1 + 1$. For $2 \leq k \in \mathbb{N}$, $\exists\ n_k \geq m_k \ni A_{n_k} \subseteq A$. Let

$$N(i) = \begin{cases} i, & \text{if } 1 \leq i \leq n_1; \\ i - n_1 + 1, & \text{if } i > n_1. \end{cases}$$

Then, $\{A_1, A_2, \ldots, A_{n_1-2}, A_{n_1-1}, A_{n_1}, A_{n_2}, A_{n_3}, \ldots\} \to A$. □

2.3. **Continuity.**

Definition 2.3.1 (F-continuity)**.** *A function f from an fts (X, τ) to an fts (Y, υ) is said to be* F-continuous *if and only if $f^{-1}\left[G^{\upsilon\text{-}open}\right]$ is τ-open $\forall\ G \in \upsilon$.*

Result 2.3.2. [2, p.187] *Composition of two F-continuous functions is again F-continuous.*

Theorem 2.3.3. [2, Theorem 4.2] *Let X and Y be fts's, and $f : X \to Y$ be a function.*

(a) *f is F-continuous.*
(b) *The f-inverse of every closed fuzzy set in Y is closed.*
(c) *For each fuzzy set $A \subseteq X$, the inverse of every nbhd of $f[A]$ is a nbhd of A.*
(d) *For each fuzzy set $A \subseteq X$ and each nbhd V of $f[A]$, there is a nbhd W of A such that $f[W] \subseteq V$.*
(e) *If $\{A_n \mid n \in \mathbb{N}\}$ be a sequence in X which converges to a fuzzy set A in X, $\{f[A_n] \mid n \in \mathbb{N}\} \to f[A]$.*

Then (a) is equivalent to (b), (c) is equivalent to (d), (a) $\implies$ (c) and (c) $\implies$ (e).

Definition 2.3.4. *A* fuzzy homeomorphism *is an F-continuous injective map from an fts X onto an fts Y, such that the inverse of the map is F-continous as well. In that case X and Y are said to be* F-homeomorphic *and each is a* fuzzy homeomorph *of the other.*

Two fts's are topologically F-equivalent if and only if they are F-homeomorphic. [Is it the same that each is coarser than the other?]

2.4. **Induced fuzzy topology.** Let (X, τ) be a crisp topological space. Let A be a fuzzy set in X.

Definition 2.4.1 (Directed set)**.** *A* directed set *$\mathcal{J}$ is a set endowed with partial order $\preceq$ such for each pair $\alpha, \beta \in \mathcal{J}$, $\exists\ \gamma \in \mathcal{J} \ni \alpha \preceq \gamma, \beta \preceq \gamma$.*

Definition 2.4.2 (Net)**.** *A* net *in X is a function $f : \mathcal{J} \to X$ where $\mathcal{J}$ is a directed set. For $\alpha \in \mathcal{J}$, $f(\alpha)$ is denoted by x_α. We denote the net itself by $(x_\alpha)_{\alpha\in\mathcal{J}}$.*

Definition 2.4.3. *A net $(x_\alpha)_{\alpha\in\mathcal{J}}$ is said to* converge *at $x \in X$ (written $x_\alpha \to x$) if for each neighbourhood U of x, $\exists\ \alpha \in \mathcal{J} \ni$*

$$\alpha \preceq \beta \implies x_\beta \in U.$$

Definition 2.4.4 (Closed fuzzy set)**.** *A is closed if whenever there exists a net $(x_\alpha)_{\alpha\in\mathcal{J}} \to x \in X$, then $\mu_A(x) \geq \limsup_{\alpha\in\mathcal{J}} \mu_A(x_\alpha)$, i.e., x is at least as much in A as the x_α ultimately are.*

But this is precisely the condition that the mapping $\mu_A : X \to \mathbb{R}$ is *upper semicontinuous*, i.e., $1 - \mu_A$, the membership function for "open" A' is *lower semicontinuous.* Thus,

Definition 2.4.5 (Induced Fuzzy Topology)**.** *An* induced fuzzy topology *on (X, τ) is the collection of all lower semicontinuous fuzzy sets in X. It is denoted by $F(\tau)$.*

Here, we are referring to the membership function as the fuzzy set. We can alternatively say, 'The collection of all fuzzy sets in X with lower semicontinuous membership functions', which is the same thing.

Proposition 2.4.6. [14, Proposition 3.2] *$F(\tau)$ thus defined will be a fuzzy topology for X.*

Thus, $(X, F(\tau))$ is an fts. Since $f : X \to \mathbb{R}$ is lower semicontinuous (respectively, upper semicontinuous) if and only if $\forall\ r \in \mathbb{R}, \{x \in X \mid f(x) \leq r\}$ is closed (respectively, $\{x \in X \mid f(x) \geq r\}$ is closed),

Proposition 2.4.7. [14, Proposition 3.3] *A fuzzy set A in X is open if and only if $\sigma_\alpha(A)$ is open $\forall\ \alpha > 0$. A is closed if and only if $\Gamma_\alpha(A)$ is closed $\forall\ \alpha > 0$.*

Notation 2.4.8. *If $S \subseteq X$ is a crisp set, we can think of its characteristic function*

$$\chi_S(x) = \begin{cases} 1 & \text{if } x \in S; \\ 0 & \text{if } x \notin S \end{cases} \qquad \forall\ x \in X$$

to represent a fuzzy set, which we shall denote by χ_S.

Result 2.4.9. [14, p. 144] $S \in \tau \implies \chi_S \in F(\tau)$.

Proposition 2.4.10 (F-continuity in induced fuzzy sets)**.** [14, Proposition 3.4] *If $(X, \tau), (Y, \upsilon)$ be topological spaces, then a mapping $T : (X, F(\tau)) \to (Y, F(\upsilon))$ is F-continuous if and only of $T : (X, \tau) \to (Y, \upsilon)$ is continuous.*

2.5. **Approaches to define compactness.** In point set topology, we had defined compactness as:

Definition 2.5.1 (Compactness as in point set topology)**.** [12] *A space X is said to be compact if every open covering $\mathcal{A}$ of X contains a finite subcollection that also covers X.*

A similar approach for fuzzy sets had been suggested in [2].

Definition 2.5.2 (Open cover for fuzzy spaces)**.** *A family $\mathcal{G}$ of open fuzzy sets is an open cover for a fuzzy set B if and only if $B \subseteq \cup_{G\in\mathcal{G}} G$. A subcover is a subfamily of $\mathcal{G}$ that is also a cover.*

Definition 2.5.3 (Fuzzy compactness, Chang)**.** *An fts (X, τ) is compact if and only if every open cover of X has a finite subcover.*

But, we can see that this definition is of no use when we deal with induced fuzzy topologies, because

Observation 2.5.4. *According to the criterion in Definition 2.5.3, no induced fts (not even if induced by a compact topological space) is compact.*

Proof. Let (Y, τ) be any fts. Even if it contains a sequence $\{A_n \mid n \in \mathbb{N}\}$ of open constant fuzzy sets such that μ_{A_n} is strictly increasing and $\mu_{A_n} \to 1$ as $n \to \infty$, then (Y, τ) cannot be compact since $\{A_n \mid n \in \mathbb{N}\}$ is an open cover for χ_Y having no finite subcover. □

Definition 2.5.5 (Alternative definition)**.** [14, Definition 3.5] *A fuzzy set A in X is compact if* $\Gamma_\alpha(A)$ *is compact* $\forall\ \alpha > 0$.

Definition 2.5.6 (FIP)**.** *A family A of fuzzy sets has the finite intersection property if and only if the intersection of the members of each finite subfamily of A is nonempty.*

2.5.1. *Results on compactness that follow from Definition 2.5.3.*

Theorem 2.5.7. [2, Theorem 5.1] *An fts is compact if and only if each family of closed fuzzy sets in it satisfying finite intersection property has a non-empty intersection.*

Theorem 2.5.8. [2, Theorem 5.2] *Let X be a compact fts, and* $f : X \to Y$ *be an F-continuous function. Then Y is compact.*

2.5.2. *Results on compactness that follow from Definition 2.5.5.*

Proposition 2.5.9. [14, Proposition 3.6] *A fuzzy set A in* $(\mathbb{R}, \mathcal{U})$ *is compact if and only if it is closed and bounded.*

Proposition 2.5.10 (An equivalent of Theorem 2.5.8 for Hausdorff codomain)**.** [14, Proposition 3.7] *Let T be a continuous mapping of* (X, τ) *into a Hausdorff space* (Y, υ)*. Then for compact* $A \subseteq X$*, TA is compact.*

2.6. **Connectedness.**

Definition 2.6.1. [14, Definition 3.8] $A \subseteq X$ *is* connected *if* $\sigma_\alpha(A)$ *is connected* $\forall\ \alpha > 0$.

Proposition 2.6.2. [14, Proposition 3.9] *Suppose* $T; (X, \tau) \to (Y, \upsilon)$ *is continuous, and* $A \subseteq X$ *be connected. Then, TA is connected.*

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