Bundles

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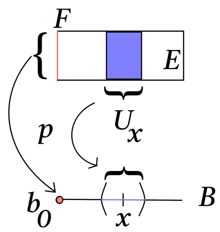
1 Fibrations

If (B, b_0) is a connected based space, a surjective continuous map $p : E \to B$ is called a *locally trivial fibration* with *fiber* F if

1. $p^{-1}(b_0) = F$;
2. every $x \in B$ has an open neighbourhood $U_x \subset B$ and a fiber-preserving homeomorphism $\psi_{U_x} : p^{-1}(U_x) \to U_x \times F$ such that the following diagram commutes:

$$\begin{array}{ccccc} E \supset & p^{-1}(U_x) & \xrightarrow[\cong]{\psi_{U_x}} & U_x \times F & \subset B \times F \\ & \downarrow p & \swarrow \pi_1 & & \\ x \in & U_x & & \subset B & \end{array}$$

The idea of a fiber bundle is something like a quotient map, loosening the condition from an equivalence relation to just a fibration map, and requiring the commutativity with that map:



The space B is called the *base space* and E is called the *total space*. This data for a *fiber bundle* is denoted by the triple (F, E, B).

A *map* or *morphism* of fiber bundles, $\Phi = (\overline{\varphi}, \varphi) : (F_1, E_1, B_1) \to (F_2, E_2, B_2)$ preserves base points and is such that the following diagram commutes

$$\begin{array}{ccc} E_1 & \xrightarrow{\overline{\varphi}} & E_2 \\ \downarrow p_1 & & \downarrow p_2 \\ B_1 & \xrightarrow{\varphi} & B_2 \end{array}$$

Such a map of fibrations determines a continuous map $\varphi_0 : F_1 \to F_2$. It is an *isomorphism* if, additionally, we have $\Phi^{-1} : (F_2, E_2, B_2) \to (F_1, E_1, B_1)$ such that $\Phi \circ \Phi^{-1} = \Phi^{-1} \circ \Phi = 1$.

- The projection map $\pi : X \times F \to X$ is the *trivial fibration* over X with fiber F.

  Any fibration that is isomorphic to the trivial fibration is trivial as well.

- Let $(1, 0)$ be the base-point of $\mathbb{S}^1 \subset \mathbb{C}$ and consider the map $f_n : \mathbb{S}^1 \to \mathbb{S}^1; z \mapsto z^n$. The fiber of f_n are the nth roots of unity.

However, this bundle is not trivial. Consider the possible map to the trivial bundle

$$\begin{array}{ccc} \mathbb{S}^1 \times F & \xrightarrow{\;\overline{\eta}\;} & \mathbb{S}^1 \\ \big\downarrow{\scriptstyle \pi_1} & & \big\downarrow{\scriptstyle f_n} \\ \mathbb{S}^1 & \xrightarrow{\;\eta\;} & \mathbb{S}^1 \end{array}$$

where $|F| = n$, so $\mathbb{S}^1 \times F$ is disconnected while $\mathbb{S}^1$ is not; so there can be no isomorphism $\overline{\eta}$ between them.

- The map $\exp : \mathbb{R} \to \mathbb{S}^1; t \mapsto e^{2\pi i t}$ is a locally trivial fibration, whose fiber is $\mathbb{Z}$. It makes the real line kind of a spiral over $\mathbb{S}^1$. It is a covering map, and hence a locally trivial fibration. This fibration is also not trivial.
- A *covering space* is a locally trivial fibration with discrete fiber.
- We can define $\mathbb{R}\mathrm{P}^n = \mathbb{S}^n / \sim$, that identifies the antipoles. The projection map $p : \mathbb{S}^n \to \mathbb{R}\mathrm{P}^n$ is a locally trivial fibration with fiber the two-point set. This is also non-trivial. The complex analogue will also work: $\mathbb{C}\mathrm{P}^n = \mathbb{S}^{2n+1} / \sim$ where $x \sim ux$ for $u \in \mathbb{S}^1 \subset \mathbb{C}$; the locally trivial fibration $p : \mathbb{S}^{2n+1} \to \mathbb{C}\mathrm{P}^n$ has fiber $\mathbb{S}^1$.
- In the Möbius band $M = [0,1] \times [0,1] / \sim$ where $(t,0) \sim (1-t,1) \forall t \in I$, let $C = \left\{ \left(\frac{1}{2}, s\right) \in M \right\}$ be the centre circle. The projection $p : M \to C; (t,s) \mapsto \left(\frac{1}{2}, s\right)$ is a locally trivial fibration with fiber $[0,1]$.

2 Smooth manifolds

Let $\mathbb{R}^n$ be the affine n-space, and any function defined one an open set $U \subset \mathbb{R}^n$ with values in $\mathbb{R}^k$ is *smooth* if its partial derivatives of all orders exist and are continuous,i.e., it is differentiable of class C^∞. In case we need to talk about an infinite-dimensional coordinate space, $\mathbb{R}^A$ can be thought of as the vector space consisting of all functions $\mathbf{x} : A \to \mathbb{R}$. $\mathbb{R}^n$ is a special case where $A = \{1, 2, \ldots, n\}$. The αth coordinate of $\mathbf{x}$ is the value of vector $x \in \mathbb{R}^A$ on $\alpha \in A$. For a function $f : Y \to \mathbb{R}^A$, the αth coordinate of $f(y)$ will be denoted by $f_\alpha(y)$. Now it is but routine to topologize $\mathbb{R}^A$ as a Cartesian product of $|A|$ copies of $\mathbb{R}$. For any subset $M \subset \mathbb{R}^A$, it has the relative topology. Thus a function $f : Y \to \mathbb{R}^A$ is continuous iff each associate function $f_\alpha : Y \to \mathbb{R}$ is continuous. For $U \subset \mathbb{R}^n$, a function $f : U \to M \subset \mathbb{R}^A$ is *smooth* if each of the associated functions $f_\alpha : U \to \mathbb{R}$ is smooth. If f is smooth then $\dfrac{\partial f}{\partial u_i}$ can be defined as a smooth function $U \to \mathbb{R}^A$ whose αth coordinate is $\dfrac{\partial f_\alpha}{\partial u_i}$ for $i \in \{1, 2, \ldots, n\}$.

A subset $M \subset \mathbb{R}^A$ is a smooth manifold of dimension $n \geq 0$ if for each $x \in M$ there exists a smooth function $h : U \to \mathbb{R}^A$ defined on an open set $U \subset \mathbb{R}^n$ such that

(i) $h : U \xrightarrow{\text{homeomorphism}} V^{\text{open}} \subset M$ where $x \in V$;

(ii) for each $u \in U$ the matrix $\left[\dfrac{\partial h_\alpha(u)}{\partial u_j}\right]$ has rank n. In other words, the vectors $\left\{\dfrac{\partial h}{\partial u_j}\right\}_{j \in [n]}$, evaluated at u, must be linearly independent. [Does this basically mean that no dimension is lost in the mapping?]

The image $h(u) = V$ of such a mapping is called a *coordinate neighbourhood* of M, and the triple (U, V, h) is called a *local parametrization* of M. The inverse $h^{-1} : V \to U \subset \mathbb{R}^n$ is called a *chart* which is a *local coordinate system* of M. The most classical and familiar examples of smooth manifolds are curves and surfaces in $\mathbb{R}^3$.

If (U, V, h) and (U', V', h') are two local parametrizations of M such that $V \cap V' \neq \phi$. Then $\varphi : (\mathbb{R}^n \supset) \left(h'\right)^{-1} \left(V \cap V'\right) \to h^{-1} \left(V \cap V'\right) \subset \mathbb{R}^n; u' \mapsto h^{-1}\left(h'(u)\right)$ is a smooth mapping. To see this, consider arbitrary $\overline{x} = h\left(\overline{u}\right) = h'\left(\overline{u'}\right) \in V \cap V'$. Choose indices $\alpha_1, \ldots, \alpha_n$ such that $\left[\dfrac{\partial h_{\alpha_i}}{\partial u_j}\right]_{n \times n}$ evaluated at $\overline{u}$

is non-singular (how am I sure that such n indices exist? Is this because no dimension is lost in the mapping for a manifold?). It follows from inverse function theorem that one can solve for $u_1, \ldots, u_n$ as smooth functions $u_j = f_j\left(h_{\alpha_1}(u), \ldots, h_{\alpha_n}(u)\right)$ for u in some neighbourhood of $\bar{u}$. This gives us $u = f\left(h_{\alpha_1}(u), \ldots, h_{\alpha_n}(u)\right)$, and setting $h(u) = h'\left(u'\right)$, it follows that the function $u' \mapsto h^{-1}h'\left(u'\right) = f\left(h'_{\alpha_1}\left(u'\right), \ldots, h'_{\alpha_n}\left(u'\right)\right)$ is smooth throughout some neighbourhood of u'.

Consider two smooth manifolds $M \subset \mathbb{R}^A$ and $N \subset \mathbb{R}^B$, and let $\bar{x} \in M$ and (U, V, h) be a local parametrization fo M with $\bar{x} = h\left(\bar{u}\right)$. A function $f : M \to N$ is said to be *smooth* at $\bar{x}$ if the composition $f \circ h : U \to N \subset \mathbb{R}^B$ is smooth throughout some neighbourhood of $\bar{u}$. This definition does not depend on the choice of local parametrization. The function $f : M \to N$ is *smooth* if it is smooth at x for every $x \in M$. It's a *diffeomorphism* if it is additionally bijective and f^{-1} is also smooth, i.e., a criterion of both-way smoothness imposed upon a homeomorphism. For a smooth manifold M, id_M is always smooth. Composition of two smooth maps $M \xrightarrow{g} M' \xrightarrow{f} M''$ is also smooth.

If M is a manifold, a *smooth path* through fixed $\bar{x} \in M$ is a smooth function $p : (-\varepsilon, \varepsilon) \to M \subset \mathbb{R}^A$ such that $p(0) = \bar{x}$ for some $\varepsilon > 0$. The *velocity vector* of such a path is defined to be $\left.\frac{dp}{dt}\right|_{t=0} = \left(\frac{dp_\alpha}{dt}(0) : a \in A\right) \in \mathbb{R}^A$. A vector $v \in \mathbb{R}^A$ is tangent to M at x if v can be expressed as a velocity vector of some smooth path through x in M. The vector v might be identified with the collection of paths p which have the common velocity vector v; this allows an intrinsic definition of tangent vector independent of the embedding in $\mathbb{R}^A$. The set of all such tangent vectors will be called the tangent space of M at x, denoted by $\mathrm{D}\, M_x$. To describe the tangent space in terms of local parametrization (U, V, h) with $h\left(\bar{u}\right) = \bar{x}$, a vector $v \in \mathbb{R}^A$ is tangent to M at $\bar{x}$ if and only if v can be expressed as a linear combination of $\left\{\frac{\partial h}{\partial u_i}\left(\bar{u}\right)\right\}_{i\in[1,n]}$. Thus, the set of all such tangent vectors, called the *tangent space* of M at x, denoted by $\mathrm{D}\, M_x$, is an n-dimensional vector space over $\mathbb{R}$. The *tangent manifold* of M is defined to be the subspace $\mathrm{D}\, M \subset M \times \mathbb{R}^A$ consisting of all pairs (x, v) with $x \in M$ and $v \in \mathrm{D}\, M_x$. As a subset of $\mathbb{R}^A \times \mathbb{R}^A$ it is a smooth manifold of dimension $2n$.

Any map $f : M \to N$ which is smooth at x determines a linear map $\mathrm{D}\, f_x$ from the tangent space $\mathrm{D}\, M_x$ to $\mathrm{D}\, N_{f(x)}$. To see this, consider in $\mathrm{D}\, M_x$, $v = \left.\frac{dp}{dt}\right|_{t=0}$, velocity vector of some smooth path $x \in M$, and define $\mathrm{D}\, f_x(v)$ to be the velocity vector $\left.\frac{d(f \circ p)}{dt}\right|_{t=0}$ of the image path $f \circ p : (-\varepsilon, \varepsilon) \to N$. This definition does not depend on the choice of p [since we can think of the velocity vector as independent of embedding of p] and $\mathrm{D}\, f_x$ is a linear mapping, called the *derivative* or the *Jacobian* of f at x. In fact, in terms of local parametrization (U, V, h) one has the explicit formula $\mathrm{D}\, f_x\left(\sum_{i=1}^{n} c_i \frac{\partial h}{\partial u_i}\right) = \sum_{i=1}^{n} c_i \frac{\partial (f \circ h)}{\partial u_i}$ for $c_i \in \mathbb{R}$.

Supposing $f : M \to N$ to be smooth everywhere, combining all the Jacobians $\mathrm{D}\, f_x$, one obtains a function $\mathrm{D} f : \mathrm{D}\, M \to \mathrm{D}\, N$ where $\mathrm{D} f(x, v) = \left(f(x), \mathrm{D}\, f_x(v)\right)$. If $\mathcal{M}\mathfrak{an}^\infty$ be the category whose objects are smooth manifolds, and morphisms smooth maps, $D : \mathcal{M}\mathfrak{an}^\infty \to \mathcal{M}\mathfrak{an}^\infty$ is a covariant functor. As a special consequence, if f is a diffeomorphism from M to N the $\mathrm{D} f$ is a diffeomorphism from $\mathrm{D}\, M$ to $\mathrm{D}\, N$.

Note that for the affine space $\mathbb{R}^n$, $\mathrm{D}\, \mathbb{R}^n_x = \mathbb{R}^n$, where in the latter case we can view $\mathbb{R}^n$ as a vector space. [WHY?] In particular, for any $u \in \mathbb{R}$, $\mathrm{D}\, \mathbb{R}_u = \mathbb{R}$. For a smooth real-valued function $f : M \to \mathbb{R}$, its derivative is $\mathrm{D}\, f_x : \mathrm{D}\, M_x \to \mathrm{D}\, \mathbb{R}_{f(x)} = \mathbb{R}$ and so $\mathrm{D}\, f_x \in \mathrm{Hom}_{\mathbb{R}}\left(\mathrm{D}\, M_x, \mathbb{R}\right)$, the dual vector space. As an element of the dual space $\mathrm{D}\, f_x = df(x)$ is called the *total differential* of f at x. From elementary calculus, we know Leibneitz rule to hold here: $\mathrm{D}(fg)_x = f(x)\,\mathrm{D}\, g_x + g(x)\,\mathrm{D}\, f_x$. If $v \in \mathrm{D}\, M_x$ (a tangent vector), $\mathrm{D}\, f_x(v) \in \mathbb{R}$ is called the *directional derivative* of the real-valued function f in the direction v. Let $C^\infty(M, \mathbb{R})$ be the vector space of all smooth real-valued functions on M. Keeping (x, v) fixed we can vary f over the vector space $C^\infty(M, \mathbb{R})$, and that gives a linear differential operator $X : C^\infty(M, \mathbb{R}) \to \mathbb{R};\ f \mapsto \mathrm{D}\, f_x(v)$. Leibnitz rule is automatically carried on: $X(fg) = f(x)X(g) + X(f)g(x)$.

One defect of the above presentation is that the "smoothness" of a manifold M is made to depend on some particular embedding of M in a coordinate space. However, we may canonically embed any smooth manifold M in one preferred coordinate space, that does not involve and specific other embedding. Let $M \subset \mathbb{R}^A$ and $F = C^\infty(M, \mathbb{R})$. Then we define $i : M \hookrightarrow \mathbb{R}^F;\ \mathbf{x} \mapsto (f(\mathbf{x})\, f \in F)$. [Seeing $\mathbb{R}^F$ as set of all functions from F to $\mathbb{R}$, each element $\mathbf{x}$ of M gives such a real-valued function on M; an element of $\mathbb{R}^F$ whose fth coordinate is given by $f(\mathbf{x})$.] Let $M_1 = i(M) \subset \mathbb{R}^F$. This M_1 is a smooth manifold in $\mathbb{R}^F$ and the canonical map $i : M \to M_1$ is a diffeomorphism. Any smooth manifold has a canonical embedding in an associated coordinate space. This suggests the following definition: Let M be a set and F be a collection of real-valued functions on M which *separates points*, i.e., for all $x \neq y \in M$ there exists $f \in F$ with $f(x) \neq f(y)$. Then M can be identified with its image under the canonical imbedding $i : M \to \mathbb{R}^F$. Basically M and M_1 are topologically same. The collection F is a *smoothness structure* on M if the subset $i(M) \subset \mathbb{R}^F$ is a smooth manifold [up to this, F is a basis for a smoothness structure] and if $F = C^\infty(M, \mathbb{R})$.