Basic Homotopy Theory, and CW Complexes: How To Build Them

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1 Homotopy

Is $\mathbb{R} \approx \mathbb{R}^2$? No, since removing one point from $\mathbb{R}$ makes it disconnected, while removing a countable number of points from $\mathbb{R}^2$ does not create a disconnection. But they are in many ways similar, and one is an expansion of the other. Homeomorphism is a very rigid condition imposed upon similarity, and does not reflect the necessary conditions for our working cases of similarity.

Problem 1 *We want a looser version of homeomorphism.*

1.1 Homotopy of maps

Definition 2 *Let $f, g : X \to Y$ be maps (continuous functions). We say $f \simeq g$ if $\exists$ map $F : X \times I \to Y \ni F(x,0) = f(x), F(x,1) = g(x) \; \forall \; x \in X$. F is a homotopy from f to g; $F : f \simeq g$.*

Observation 3 *"$\simeq$" is an equivalence relation. Let $f \in C(X,Y)$ be a map from X to Y; $[f] = \{g \in C(X,Y) \mid f \simeq g\}$, equivalence class of f.*

Proof.

(i) For any map $f : X \to Y$, $f \simeq f$.

Here we simply consider $F(x,t)$ to be independent of t, and $F(x,t) = f(x) \; \forall \; x \in X$.

(ii) Let $f, f' : X \to Y$. Then, $f \simeq f' \Rightarrow f' \simeq f$.

There exists homotopy $F : X \times I \to Y$ such that $F(x,0) = f(x)$ and $F(x,1) = f'(x)$. We define $G : X \times I \to Y$ such that $G(x,t) = F(x, 1-t) \; \forall \; t \in [0,1]$. We have that G is a homotopy between f' and f.

(iii) Let $f, f', f'' : X \to Y$. Then, $f \simeq f', f' \simeq f'' \Rightarrow f \simeq f''$.

There exists $F : X \times I \to Y$ such that $F(x,0) = f(x),\; F(x,1) = f'(x)$ and $F' : X \times I \to Y$ such that $F'(x,0) = f'(x),\; F'(x,1) = f''(x)$. Define $G : X \times I \to Y$ with

$$G(x,t) = \begin{cases} F(x,2t) & \text{for } t \in \left[0, \frac{1}{2}\right], \\ F'(x, 2t-1) & \text{for } t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

We note that G is well-defined and for $t = \dfrac{1}{2}$, $F(x,2t) = f'(x) = F'(x, 2t-1)$. Because G is continuous on the two closed subsets $X \times \left[0, \dfrac{1}{2}\right]$ and $X \times \left[\dfrac{1}{2}, 1\right]$ of $X \times I$, by *Pasting Lemma*, G is continuous throughout and the required homotopy.$\square$

1.2 Homotopy equivalence of spaces

Definition 4 *Suppose* $\exists\ f : X \to Y, g : Y \to X \ \ni\ g \circ f \simeq id_X, f \circ g \simeq id_Y$. *Then we say that* X *is homotopically equivalent to* Y*,* $X \simeq Y$*, or that* X *and* Y *have the same homotopy type.*

Observation 5 $X \approx Y \Rightarrow X \simeq Y$

Proof. f^{-1} exists and is continuous. $g = f^{-1}$.

Remark 6 *Converse is not true.*

Counterexample. $X = \mathbb{R}^2$, $Y = \{(0,0)\}$. $X \not\approx Y$ since X is not compact but Y is. Let $F : \mathbb{R}^2 \times I \to \mathbb{R}^2$; $\left(x = \left(x_1, x_2\right), t\right) \mapsto (1-t)\,x \ \left[x \in \mathbb{R}^2, t \in [0,1]\right]$. Then, $F(x,0) = \mathrm{id}_X$ and $F(x,1) = \mathrm{id}_Y$. So, there exists a homotopy from id_X to id_Y.

Alternatively, we define $i : \{(0,0)\} \to \mathbb{R}^2, (0,0) \mapsto (0,0)$; $c : \mathbb{R}^2 \to \{(0,0)\}$. Then $i \circ c : \mathbb{R}^2 \to \mathbb{R}^2; x \mapsto (0,0)\ \forall x \in \mathbb{R}^2$ and $c \circ i : \{(0,0)\} \to \{(0,0)\}$; $c \circ i = \mathrm{id}_Y$.□

Definition 7 *A path-connected space is called* contractible *is it is homotopically equivalent to a one-point space, i.e., it has the same homotopy type as a point.*

Observation 8 *Any 2 maps* $f, g : X \to Y$*, where* Y *is contractible, are homotopic.*

Proof. Let Y be contractible. So there should exist a homotopy $H : 1_Y \simeq f_c$ where $f_c : Y \to \{c\} \subseteq Y$.

Let X be another space and we consider a map $g : X \to Y$. Then $H \circ g : X \times I \to Y$ is a homotopy between g and the constant map $\tilde{f} : X \to \{c\} \subseteq Y$. So every map $g : X \to Y$ is homotopic to the constant map $\tilde{f}$. Since homotopy is an equivalent relation, this implies that all maps $X \to Y$ are homotopic.□

Definition 9 *A* retraction *is a map* r *from a space* X *to its subset* A*, such that* $r|_A = id_A$.

A deformation retraction is a homotopy $F : X \times I \to A \subseteq X$ *such that* $F|_{A,I} = id_A$. *Here, additionally, the map* F *is continuous. The retract is said to be "relative* A*" or* $rel A$.

1.3 Homotopy between Paths

For a topological space X, a *path* is a map $\sigma : [0,1] \to X$. A path is called a *loop* at $x \in X$ if $\sigma(0) = \sigma(1) = x$.

Let α and β be paths from x to y in X. We say that α *is homotopic to* β relative to the endpoints $(0,1)$

$$\alpha \simeq \beta \ \mathrm{rel}(0,1)$$

if $\exists\ F : I \times I \to X \ \ni$

$$\begin{aligned} F(s,0) = \alpha(s) \quad &\forall \quad s \in I;\\ F(s,1) = \beta(s) \quad &\forall \quad s \in I;\\ F(0,t) = \alpha(0) = \beta(0) = x \quad &\forall \quad t \in I;\\ F(1,t) = \alpha(1) = \beta(1) = y \quad &\forall \quad t \in I. \end{aligned}$$

We define $F_t : I \to X$; $F_t(s) = F(s,t)$ for each fixed $t \in I$. Therefore, F_t is also a path from x to y. Thus, homotopy can be seen as kind of a continuous deformation of path α to path β.

We define $P(X; x, y) = \{\alpha : I \to X \mid \alpha(0) = x, a(1) = y\}$. "Being homotopic" is an equivalence relation on $P(X; x, y)$ because

(i) Consider $F(s,t) = \alpha(s) \forall\ s, t \in I$ which shows that $\alpha \simeq \alpha$ rel$(0,1)$, i.e., reflexivity.

(ii) If $F : \alpha \simeq \beta$ rel$(0,1)$ then $G : I \times I \to X$; $G(s,t) = F(s, 1-t)$ gives $G : \beta \simeq \alpha$ rel$(0,1)$. Symmetry.

(iii) If we have $F : \alpha \simeq \beta$ rel$(0,1)$ and $G : \beta \simeq \gamma$ rel$(0,1)$, then we define the *concatination* of F and G by

$$(F * G)(s,t) = \begin{cases} F(s,2t) & \text{for } 0 \le t \le \frac{1}{2};\\ G(s,2t-1) & \text{for } \frac{1}{2} \le t \le 1. \end{cases}$$

which is continuous by the pasting lemma, whence $F * G : \alpha \simeq \gamma$ rel$(0,1)$. Transitivity.

Note 10 *Concatination refers to basically traversing the first path in half the time with double speed, and then the second path in the other half of the time with double speed.*

1.4 Fundamental Group

The equivalence class of $\alpha \in P(X; x, y)$ is denoted by $[\alpha]$ by $[\alpha] = \left\{\alpha' \in P(X; x, y) \mid \alpha \simeq \alpha' \text{ rel}(0,1)\right\}$. The set of all such equivalence classes is denoted by $P[X; x, y]$. Let's divert our discussion to loops at a point, say, x_0. We denote $P\left[X; x_0, x_0\right]$ by $\pi_1\left(X, x_0\right) \coloneqq \left\{[\sigma] \mid \sigma \text{ is a loop at } x_0 \in X\right\}$. We define a binary operation $[*] : \pi_1\left(X, x_0\right) \times \pi_1\left(X, x_0\right) \to \pi_1\left(X, x_0\right);\ [\sigma][*][\tau] \mapsto [\sigma * \tau]$, where concatination $\sigma * \tau : I \to X$ is defined in our old way:

$$\sigma * \tau(s) = \begin{cases} \sigma(2s) & \text{when } 0 \le s \le \frac{1}{2}; \\ \tau(2s-1) & \text{when } \frac{1}{2} \le s \le 1. \end{cases}$$

The operation $[*]$ is well-defined, i.e., it does not depend on the choice of representatives σ and τ. If $\sigma \simeq \sigma'$ rel$(0,1)$ and $\tau \simeq \tau'$ rel$(0,1)$ then $\sigma * \tau \simeq \sigma' * \tau'$ rel$(0,1)$. Now we observe that

(i) Of course, concatination gives another loop, and hence its equivalence class gives an element of $\pi_1\left(X, x_0\right)$.

(ii) $*$ is associative and hence so is $[*]$.

(ii) $[e]$ is the unit in $\pi_1\left(X, x_0\right)$ where $e : I \to X;\ e(s) = x_0\ \forall\, s \in I$, the *constant loop* at x_0.

(iii) We observe that $\sigma^{-1}(s) = \sigma(1-s)$; $[\sigma]^{-1} = \left[\sigma^{-1}\right]$ where $\sigma * \sigma^{-1} \simeq \sigma^{-1} * \sigma \simeq e$ rel$(0,1)$.

We conclude that $\pi_1\left(X, x_0\right)$ forms a group with $[*]$ as a binary operation, known as the *fundamental group*.

If $x_1 \in X$ is another point in X then $\pi_1\left(X, x_1\right)$ makes sense.

Theorem 11 *If X is path-connected then a path α from x_0 to x_1 induces an isomorphism α_* from $\pi_1\left(X, x_0\right)$ to $\pi_1\left(X, x_1\right)$ by*

$$\begin{array}{rcl} \alpha_* : \pi_1\left(X, x_0\right) & \to & \pi_1\left(X, x_1\right); \\ [\sigma] & \mapsto & \left[\alpha^{-1} * \sigma * \alpha\right];\ \textit{describe } \alpha^{-1}\textit{, loop } \sigma\textit{, return to } x_1. \end{array}$$

To check that α_* is a well-defined isomorphism, we need to show that

(i) $\alpha_*([\sigma][\tau]) = \alpha_*([\sigma])\,\alpha_*([\tau])$;

(ii) $\left(\alpha_*\right)^{-1} = \left(\alpha^{-1}\right)_*$.

Because the operation $*$ is well-defined, if σ is a loop based at x_0, then $\alpha^{-1} * (\sigma * \alpha)$ is a loop based at x_1. To show α_* to be a group isomorphism, we need

$$\begin{array}{rcl} \alpha_*([\sigma])\,\alpha_*([\tau]) & = & \left(\left[\alpha^{-1} * \sigma * \alpha\right]\right)\left(\left[\alpha^{-1} * \tau * \sigma\right]\right) \\ & = & \left[\alpha^{-1} * \sigma * \alpha * \alpha^{-1} * \tau * \sigma\right] \\ & = & \left[\alpha^{-1}\right][\sigma * \tau][\alpha] \\ & = & \alpha_*([\sigma][\tau]). \end{array}$$

Let β denote the path α^{-1}, then β_* is the inverse for α_*. For each $[\sigma] \in \pi_1\left(X, x_1\right)$, $\beta_*([\sigma]) = \left[\beta^{-1} * \sigma * \beta\right] = \left[\alpha * \sigma * \alpha^{-1}\right] \Rightarrow \alpha_*\left(\beta_*[\sigma]\right) = \left[\alpha^{-1} * \alpha * \sigma * \alpha^{-1} * \alpha\right] = [\sigma]$. Similarly $\beta_*\left(\alpha_*[\sigma]\right) = [\sigma]$ for all $[\sigma] \in \pi_1\left(X, x_0\right)$.

Moreover, if $\alpha \simeq \beta$ rel$(0,1)$ for $\alpha, \beta \in P\left[X; x_0, x_1\right]$, then $\alpha_* = \beta_*$.

Thus, it makes sense to talk about the fundamental group of a space, irrespective of the base point.

1.5 Homotopy groups

Let $f : X \to Y$, and $[f]$ be the equivalence class containing f, called its *homotopy class*. The collection of all homotopy classes $X \to Y$ is denoted by $[X, Y]$. In the set $\pi_n(X) = [S^n, X]$, we can similarly define a binary operation for $f, g : S^n \to X$,

$$(fg)\left(s_1,\ldots,s_n\right)=\begin{cases} f\left(2s_1,s_2,\ldots,s_n\right),\ s_1\in\left[0,\frac{1}{2}\right],\\ g\left(2s_1-1,s_2,\ldots,s_n\right),\ s_1\in\left[\frac{1}{2},1\right].\end{cases}$$

The identity will be the class of the constant map to the base point, and $-f\left(s_1,s_2,\ldots,s_n\right)=f\left(1-s_1,s_2,\ldots,s_n\right)$. This group $\pi_n(X)$ is called the *nth (ordinary) homotopy group* of X. In fact, for $n\geq 2$ they are all abelian.

1.6 Homotopy extension property

2 Cell complexes

2.1 Weak Topology

Let X be a topological space and let $W_\alpha, \alpha\in A$ be subsets of X that in themselves are topological spaces. A weak topology on X is the largest topology such that the inclusion map $W_\alpha\hookrightarrow X$ is continuous for each $\alpha\in A$. Equivalently, a subset U of X is open iff $U\cap W_\alpha\subseteq W_\alpha$ is open for each $\alpha\in A$. The collection $\left\{U\subseteq X \mid U\cap W_\alpha\subseteq_{\text{open}} W_\alpha\ \forall\,\alpha\in A\right\}$ forms a topology on X, and every other topology that has the inclusion maps continuous, is contained in this weak topology.

Given $\left\{W_\alpha\right\}_{\alpha\in A}$ we can, if they are not disjoint, consider homeomorphic spaces that are disjoint, and take disjoint union $\coprod_{\alpha\in A} W_\alpha=X$, on which we can consider the weak topology.

2.2 Identification maps

2.3 Adjunction spaces

Let X and Y be unbased topological spaces. We consider $W\subset X$ and let $\varphi: W\to Y$ be a free map, i.e., a map which need not map base point to base point. On $Y\sqcup X$ we define the equivalence relation $w\sim\varphi(w)$. This gives us the identification space $Y\cup_\varphi X$, called the *adjunction space*.

2.4 Wedge

Let $\left(X_\alpha,x_\alpha\right);\alpha\in A$ be based spaces. When we quotient the disjoint union $\coprod X_\alpha$ by $\left\{x_\alpha \mid \alpha\in A\right\}$, we basically identify all the base points into one, and this gives us the *wedge* of $\left\{\left(X_\alpha,x_\alpha\right)\right\}_{\alpha\in A}$, denoted as $\bigvee_{\alpha\in A} X_\alpha$.

2.5 CW complexes

Note that for each $\alpha\in A$, there is an injection $i_\alpha: X_\alpha\to\bigvee_{\alpha\in A} X_\alpha; x\mapsto[x]\ \forall\,x\in X_\alpha$. Furthermore, the maps $f_\alpha: X_\alpha\to Y$ determine a unique map $f:\bigvee_{\alpha\in A} X_\alpha\to Y\ \ni\ f i_\alpha=f_\alpha$.

If we consider X to be an unbased space and assume $\varphi_\alpha: S^{n-1}=S^{n-1}_\alpha\to X$ to be free maps for $\alpha\in A$, these determine a free map (by pasting lemma) $\varphi:\coprod_{\alpha\in A} S^{n-1}_\alpha\to X$. Let $E^{n-1}_\alpha=E^n$ for all $\alpha\in A$. Since $\coprod_{\alpha\in A} S^{n-1}_\alpha\subset\coprod_{\alpha\in A} E^n_\alpha$, we can form the adjunction space by quotienting with respect to $\coprod_{\alpha\in A} S^{n-1}_\alpha$ as: $X\cup_{\alpha\in A}\coprod_{\alpha\in A} E^n_\alpha$. This procedure is called *attaching n-cells*.

We consider an unbased Hausdorff topological space X, which we construct as follows: we consider a chain of subsets $X^0\subseteq X^1\subseteq X^2\subseteq\cdots\subseteq X^{n-1}\subseteq X^n\subseteq\cdots\subseteq X$, which we construct by considering X^0 as a 0-cell or consisting of discrete vertices, and attaching n-cells to X^{n-1} to form X^n. We have to assume that for each n, there exist *attaching maps* $\varphi_\beta: S^{n-1}\to X^{n-1};\beta\in B$ and we thus form $X^n\coloneqq X^{n-1}\cup_\varphi\coprod_{\beta\in B} E^n_\alpha$. Note that with each such formation of the disjoint union $X^{n-1}\sqcup\coprod_{\beta\in B} E^n_\alpha$ and subsequent quotienting, for each $\beta\in B$, we get a continuous *characteristic function* $\Phi_\beta : \left(E^n, S^{n-1}\right) \to \left(X^n, X^{n-1}\right)$ and $\Phi|_{S_\beta^{n-1}} = \varphi_\beta$. The image $\Phi_\beta\left(E_\beta^n - S_\beta^{n-1}\right) = e_\beta^n$ is called an *open n-cell* (although it need not be open in X, and $\Phi_\beta|_{E_\beta^n - S_\beta^{n-1}} : E_\beta^n - S_\beta^{n-1} \to e_\beta^n$ is a homeomorphism. The closure of e_β^n is $\Phi_\beta\left(E_\beta^n\right)$ and the topological boundary $\partial e_\beta^n = \Phi_\beta\left(S_\beta^{n-1}\right)$. Then $\Phi_\beta : \left(E^n, S^{n-1}\right) \to \left(\overline{e}_\beta^n, \partial e_\beta^n\right)$ is a continuous function of pairs. This closed set $\overline{e}_\beta^n = \Phi(E^n) \subseteq X^n \subseteq X$ is called a *(closed) n-cell* that is being attached. X as a whole is endowed with the weak topology with respect to all the closed cells $\left\{\overline{e}_\beta^n\right\}_{\beta\in B, n\in\mathbb{N}\cup\{0\}}$. Here, $X^n - X^{n-1}$ is a disjoint union of the e_β^n and we can write $X^n = X^{n-1} \cup \bigcup_{\beta\in B} e_\beta^n$ or $X^n = X^{n-1} \cup \bigcup_{\beta\in B} \overline{e}_\beta^n$.

A CW complex is called finite if it has finitely many cells, and is called finite-dimensional (dimension N) if there exists $N \in \mathbb{N}$ such that $X^{N-1} \neq X$ but $X^n = X$ for all $n \geq N$.